Feuille 8

1) $k'_{\parallel} = \gamma (k_{\perp} - v\, \omega_k)$

$$\omega' = \gamma(\omega - v\, k_{\parallel})$$

$$dk'_{\parallel} = \gamma \left(dk_{\perp} - v \frac{\vec{k}\cdot d\vec{k}}{\omega_k} \right) = \gamma\, dk_{\parallel} \left(1 - v \frac{k_{\parallel}}{\omega_k}\right) + \gamma v \frac{\vec{k}_{\perp} d\vec{k}}{\omega_k}$$

$$dk'^3 = d^3k\, \gamma \left(1 - \frac{v k_{\parallel}}{\omega_k}\right)$$

$$\frac{d^3k'}{\omega_{k'}} = \frac{d^3k}{\omega_k} \quad \text{invariant}$$

$\hookrightarrow d\tilde{k}$ scalaire

2) $\langle q | q' \rangle = F^*(q) F(q') \langle 0 | a_q a^\dagger_{q'} | 0 \rangle$

$$= (2\pi)^3 (2\omega_q)\, \delta(q - q')$$

$$F(q) = \frac{1}{(2\pi)^{3/2} \sqrt{2\omega_q}} \qquad |q\rangle = \int \frac{d^3q}{(2\pi)^{3/2}\sqrt{2\omega_q}}\, a^\dagger_q |0\rangle$$

3) $|r\rangle = \alpha \int \frac{d^3q}{(2\pi)^3}\, e^{-iqr}\, |q\rangle$

$$\langle r' | r \rangle = \alpha^2 \int \frac{d^3q\, d^3q'}{(2\pi)^6}\, e^{-i(-q'r' + qr)}\, \delta(q' - q)$$

$$= \frac{\alpha^2}{(2\pi)^3}\, \delta(r' - r)$$

on choisit $\alpha = (2\pi)^{3/2}$

$$|r\rangle = \int \frac{d^3q}{(2\pi)^{3/2}} e^{-iqr} |q\rangle$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3q}{\sqrt{2\omega_q}} e^{-iqr} a_q^\dagger |0\rangle$$

$$\hookrightarrow \langle r'|r\rangle = \delta(r'-r)$$

3] $$|R\rangle = \alpha \int d^3r \; e^{-\frac{r^2}{2R^2}} |r\rangle$$

$$\langle R|R\rangle = \alpha^2 \int d^3r \; e^{-\frac{r^2}{R^2}} = \alpha^2 (\pi R^2)^{3/2}$$

$$\hookrightarrow \alpha = \frac{1}{(\pi R^2)^{3/4}} \qquad f(r) = \frac{e^{-\frac{r^2}{2R^2}}}{(\pi R^2)^{3/4}}$$

$$\langle R|N|R\rangle = \int d^3r \, d^3r' \, f(r) f(r') \langle r'|N|r\rangle$$

$$\langle r'|N|r\rangle = \frac{1}{(2\pi)^6} \int \frac{d^3q \, d^3q' \, d^3k}{\cancel{2}\sqrt{\omega_q \omega_{q'}}} e^{i(q'r'-qr)} \langle 0|a_{q'} a_k^\dagger a_k a_q^\dagger|0\rangle$$

$$= \frac{1}{\cancel{(2\pi)^6}} \int \frac{d^3q \, d^3q' \, d^3k}{\cancel{4}\cancel{\sqrt{\omega_q \omega_{q'}}\,\omega_k} (2\pi)^3} e^{i(q'r'-qr)} \cancel{(2\pi)^6} \cancel{4\omega_q}\, \omega_q \, \delta(q-k) \, \delta(q'$$

$$= \delta(r'-r).$$

$$\langle R|N|R\rangle = \int d^3r \, f(r)^2 = 1$$

$$\langle r'|N^2|r\rangle = \int d^3r'' \langle r'|N|r''\rangle\langle r''|N|r\rangle = \delta(r'-r).$$

$|\ell\rangle$ s'interpreta come un stato di 1 particella?

SJ

$$\langle \ell | H | \ell \rangle = \int d^3r\, d^3r'\, f(r') f(r) \langle r' | H | r \rangle .$$

$$\langle r' | H | r \rangle = \frac{1}{(2\pi)^6} \int \frac{d^3q\, d^3q'\, d^3k}{2\sqrt{\omega_q \omega_{q'}}} \omega_k\, e^{i(q'r' - qr)} \langle 0 | a_{q'} a^\dagger_k a_k a^\dagger_q | 0 \rangle$$

$$= \int \frac{d^3q\, d^3q'\, d^3k}{(2\pi)^3 \cancel{2} 4\sqrt{\omega_q \omega_{q'}}} e^{i(q'r' - qr)} (2\omega_k)^2 \delta(k - q') \delta(k - q) .$$

$$= \int \frac{d^3k}{(2\pi)^3} e^{ik(r' - r)} \omega_k .$$

$$\langle \ell | H | \ell \rangle = \int \frac{d^3k}{(2\pi)^3} \left[\int d^3r \frac{e^{-ikr - \frac{r^2}{2\ell^2}}}{(\pi\ell^2)^{3/4}} \right] \left[\int d^3r' \frac{e^{ikr' - \frac{r'^2}{2\ell^2}}}{(\pi\ell^2)^{3/4}} \right] \omega_k$$

$$\int d^3r\, e^{-\frac{1}{2\ell^2}\left\{ (r + ik\ell^2)^2 + k^2\ell^4 \right\}} = (2\pi\ell^2)^{3/2} e^{-\frac{k^2\ell^2}{2}}$$

$$\langle \ell | H | \ell \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{(2\pi\ell^2)^3}{(\pi\ell^2)^{3/2}} \omega_k\, e^{-k^2\ell^2}$$

$$= \frac{\cancel{8}\ell^3}{\pi^{3/2}} \int \frac{d^3k}{\cancel{(2\pi)^3}} \sqrt{k^2 + m^2}\, e^{-k^2\ell^2}$$

$\vec{K} = \ell \vec{k}$

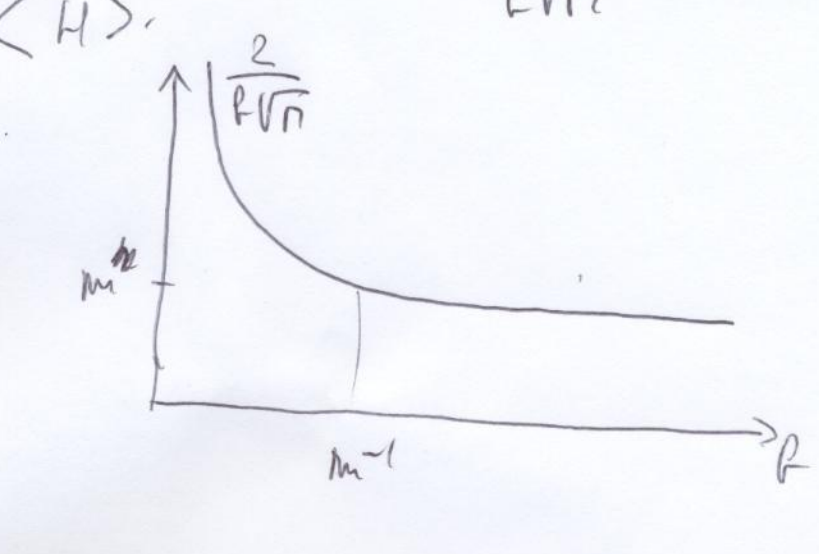
$$\langle l|H|l\rangle = \frac{1}{l\,\pi^{3/2}}\int d^3k\,\sqrt{k^2+l^2m^2}\;e^{-k^2}$$

Si $lm \gg 1$

$$\langle l|H|l\rangle = \frac{m}{\pi^{3/2}}\int d^3k\,e^{-k^2} = m.$$

Si $lm \ll 1$

$$\begin{aligned}\langle l|H|l\rangle &= \frac{1}{l\,\pi^{3/2}}\int d^3k\,k\,e^{-k^2}\\ &= \frac{1}{l\,\pi^{3/2}}\,4\pi\int_0^\infty k^3\,dk\,e^{-k^2}\\ &= \frac{4}{l\sqrt{\pi}}\int_0^\infty \frac{dx}{2}\,x\,e^{-x}\\ &= \frac{2}{l\sqrt{\pi}}\,\Gamma(2) = \frac{2}{l\sqrt{\pi}}.\end{aligned}$$



Si on localise trop $|l\rangle$, l'énergie n'est plus celle d'une particule de masse m dans son référentiel au repos $\Rightarrow$ plus aucune interprétation en terme de particule.

$$\langle r'|H^2|r\rangle = \int d^3r'' \,\langle r'|H|r''\rangle\langle r''|H|r\rangle.$$

$$= \int d^3r'' \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\, e^{ik(r-r'')}\, e^{ik'(r''-r')}\, \omega_k\omega_{k'}$$

$$= \int \frac{d^3k}{(2\pi)^3}\, e^{ik(r-r')}\, \omega_k^2$$

$$\langle b|H^2|b\rangle = \frac{b^3}{\pi^{3/2}} \int d^3k\, (k^2+m^2)\, e^{-k^2b^2}$$

$$= \frac{1}{b^2\pi^{3/2}} \int d^3K\, (K^2+m^2b^2)\, e^{-K^2}$$

$$= \frac{4\pi}{b^2\pi^{3/2}} \int K^2\, dK\, (K^2+m^2b^2)\, e^{-K^2}$$

$$= \frac{4\pi}{b^2\cancel{\sqrt{\pi}}}\left(m^2b^2\frac{\cancel{\sqrt{\pi}}}{4} + \frac{3}{8}\cancel{\sqrt{\pi}}\right).$$

$$= m^2 + \frac{3}{2b^2}$$

$\hookrightarrow \sim m^2$ if $mb \gg 1$

$\sim \frac{3}{2b^2}$ if $mb \ll 1$.

6) $\int d^3x \, \mathcal{N}(x) = \int \tilde{dk}\, \tilde{dk'}\, 2\omega_k a^\dagger_k a_k (2\pi)^3 \delta(k'-k)$

$= \int \tilde{dk}\, a^\dagger_k a_k = N$

$\int d^3x \, \mathcal{H}(x) = \int \tilde{dk}\, \omega_k a^\dagger_k a_k = H$

7) $\langle r | \mathcal{N}(x) | r' \rangle = \frac{2}{(2\pi)^6} \int \frac{d^3q\, d^3q'}{2\sqrt{\omega_q \omega_{q'}}} \tilde{dk}\, \tilde{dk'} \sqrt{\omega_k \omega_{k'}}\, e^{i(k'-k)x}\, e^{i(q\ldots}$

$\langle 0 | a_q a^\dagger_k a_{k'} a^\dagger_{q'} | 0 \rangle$

$= \frac{1}{(2\pi)^6} \int d^3q\, d^3q'\, e^{iq(r-x)}\, e^{iq(x-r')}$

$= \delta(r-x)\,\delta(r'-x)$

↳ conforte notre interprétation de $|r\rangle$ comme ket d'une particule localisée en $\vec{r}$.

8) $\langle r | \mathcal{H}(x) | r' \rangle = \frac{2}{(2\pi)^6} \int \frac{d^3q\, d^3q'}{2\sqrt{\omega_q \omega_{q'}}} \tilde{dk}\, \tilde{dk'}\, \omega_k \omega_{k'}\, e^{i(k'-k)x}\, e^{i(qr-q'r')}$

$\langle 0 | a_q a^\dagger_k a_{k'} a^\dagger_{q'} | 0 \rangle$

$= \frac{1}{(2\pi)^6} \int d^3q\, d^3q'\, \sqrt{\omega_q}\, \sqrt{\omega_{q'}}\, e^{iq(r-x)}\, e^{iq(x-r')}$.

$F(m^2, x) = \int d^3q\, (q^2+m^2)^{\frac{1}{4}}\, e^{i\vec{q}\vec{x}}$

$= 2\pi \int_0^\infty q^2 dq \int_{-1}^{1} d\cos\theta\, (q^2+m^2)^{\frac{1}{4}}\, e^{iqx\cos\theta}$

$$= 2\pi \int q^2 dq \, (q^2+m^2)^{\frac{1}{4}} \left[\frac{e^{iqx\cos\theta}}{iqx} \right]_{-1}^{1}$$

$$= \frac{4\pi}{x} \int q\,dq \, (q^2+m^2)^{\frac{1}{4}} \sin qx.$$

$$\partial_{m^2} F(m^2, x) = \frac{4\pi}{x} \int_0^\infty \frac{q}{(q^2+m^2)^{3/4}} \sin qx. \qquad \underset{q\to\infty}{\sim} \frac{q \sin qx}{q^{3/2}} \quad \text{CV}.$$

intégrale où $\sin qx \to \underbrace{e^{iqx}}_{\to 0}$ si $q \to i\underbrace{\alpha}_{\to +\infty}$

↳ l'intégrale est dominée par la singularité à $q = im$.

↳ $\partial_{m^2} F \sim e^{-mx}$.

↳ $F \sim e^{-mx}$.

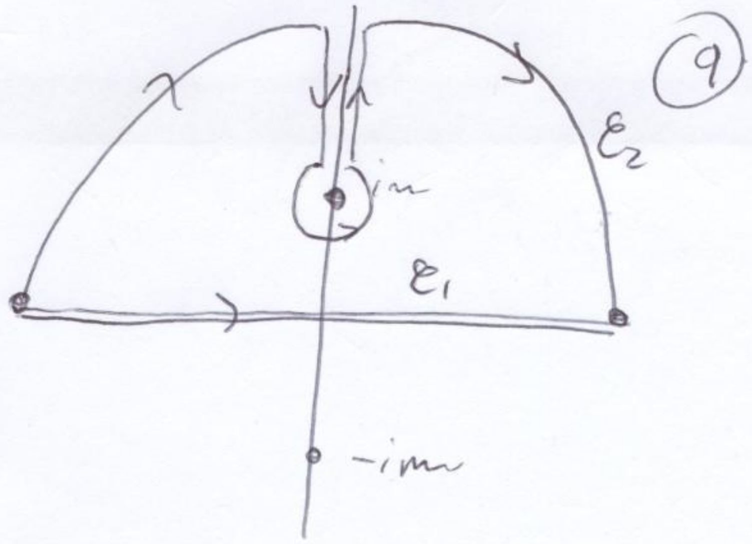
En fait, cette intégrale s'exprime en fonction de la fonction de Bessel $K_{\frac{3}{4}}$ dont on connaît le comportement asymptotique, et on retrouve le comportement précédent...

Pour faire proprement le calcul :

$$I(x) = \int_0^\infty \frac{q}{(q^2+m^2)^{3/4}} \sin qx$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{q}{(q^2+m^2)^{3/4}} e^{iqx}$$

Pour faire proprement le calcul : (Claude).



sur $\mathcal{E}_2$, l'intégrale à $|q|=\infty$ est nulle.

$$I(x) = \frac{i}{2}\int_\infty^m \frac{y\,dy}{(m^2+y^2)^{3/4}\, e^{-\frac{i3\pi}{4}}} e^{-yx} + i\int_m^\infty \frac{y\,dy}{(-m^2+y^2)^{3/4}\, e^{\frac{i3\pi}{4}}} e^{-yx}$$

$$+ \, O(\varepsilon^{1/2}) \to 0$$

$$I(x) = -\frac{e^{-\frac{i3\pi}{4}} - e^{\frac{i3\pi}{4}}}{2i}\int_m^\infty \frac{y\,dy}{(-m^2+y^2)^{3/4}} e^{-y}$$

$x = y - m$

$y = x + m$

$$= \frac{\sqrt{2}}{2} e^{-mx}\int_0^\infty \frac{\alpha+m}{(\alpha(\alpha+2m))^{3/4}} e^{-x\alpha}\, d\alpha$$

$x\alpha = \ell$

$$= \frac{e^{-mx}}{\sqrt{2}}\int_0^\infty \frac{\frac{\ell}{x}+m}{\left(\frac{\ell}{x}\left(\frac{\ell}{x}+2m\right)\right)^{3/4}} e^{-\ell}\, \frac{d\ell}{x}$$

$$= \frac{e^{-mx}}{\sqrt{2}} \frac{x^{3/2}}{x^2}\int_0^\infty \frac{\ell+mx}{(\ell(\ell+2mx))^{3/4}} e^{-\ell}\, d\ell$$

$$\simeq \frac{e^{-mx}}{\sqrt{2x}} \frac{mx}{(2mx)^{3/4}}\int_0^\infty \frac{e^{-\ell}\, d\ell}{\ell^{3/4}} \simeq \frac{e^{-mx}}{\sqrt{2x}\, 2^{3/4}} (mx)^{\frac{1}{4}}\, \Gamma\left(\frac{1}{4}\right)$$

$$\simeq \left(\frac{m}{x}\right)^{\frac{1}{4}} e^{-mx}$$

$\mathcal{I}(\pi)$

$$\langle n | H(n) | n \rangle \approx e^{-2m(n-x)}$$

g)

$$|q, q'\rangle = \alpha \, a^{\dagger}_{q} \, a^{\dagger}_{q'} |0\rangle$$

$$\langle q_1 q'_1 | q_2 q'_2 \rangle = \alpha^2 \langle 0 | a_{q'_1} a_{q_1} a^{\dagger}_{q_2} a^{\dagger}_{q'_2} | 0 \rangle$$

$$= \alpha^2 \Big\{ \langle 0 | a_{q'_1} a^{\dagger}_{q_2} a_{q_1} a^{\dagger}_{q'_2} | 0 \rangle + (2\pi)^3 2\omega_{q_1} \delta(q_1 - q_2) \langle 0 | a_{q'_1} a^{\dagger}_{q'_2} | 0 \rangle$$

$$= \alpha^2 \Big\{ (2\pi)^3 2\omega_{q_1} \delta(q_1 - q'_2) \langle 0 | a_{q'_1} a^{\dagger}_{q_2} | 0 \rangle + (2\pi)^6 4 \omega_{q_1} \omega_{q'_1} \delta(q_1 - q_2) \delta(q'_1 - q'_2) \Big\}$$

$$= \alpha^2 (2\pi)^6 4 \omega_{q_1} \omega_{q'_1} \Big\{ \delta(q_1 - q'_2) \delta(q'_1 - q_2) + \delta(q_1 - q_2) \delta(q'_1 - q'_2)$$

o vale que $\int d^3 q_2 \, d^3 q'_2 \langle q_1 q'_1 | q_2 q'_2 \rangle = 1$

$\hookrightarrow \langle q_1 q'_1 | q_2 q'_2 \rangle = \frac{1}{2} \Big\{ \delta(q_1 - q_2) \delta(q'_1 - q'_2) + \delta(q_1 - q'_2) \delta(q_2 - q'_1) \Big\}$

$$\hookrightarrow \alpha = \frac{1}{\sqrt{2} \, (2\pi)^3 \, 2\sqrt{\omega_{q_1} \omega_{q'_1}}}$$

$$|q q'\rangle = \frac{1}{\sqrt{2} (2\pi)^3 2 \sqrt{\omega_{q_1} \omega_{q'_1}}} \, a^{\dagger}_{q} a^{\dagger}_{q'} |0\rangle$$

$$|n n'\rangle = \int \frac{dq \, dq'}{(2\pi)^3} e^{-i(qn + q'n')} |q q'\rangle.$$

$$\langle r_1 r_1' \mid r_2 r_2' \rangle = \int \frac{dq_1 dq_1' dq_2 dq_2'}{(2\pi)^6} e^{i(q_1 r_1 + q_1' r_1' - q_2 r_2 - q_2' r_2')} \langle q_1 q_1' \mid q_2 q_2' \rangle$$

$$= \frac{1}{2} \int \frac{d q_1 d q_2}{(2\pi)^6} \left\{ e^{i q_1 (r_1 - r_2) + i q_2 (r_2 - r_2')} + e^{i q_1 (r_1 - r_2') + i q_2 \ldots} \right.$$

$$= \frac{1}{2} \left\{ \delta(r_1 - r_2)\delta(r_1' - r_2') + \delta(r_1 - r_2')\delta(r_1' - r_2) \right\}$$

$$|\Psi\rangle = \int d^3 r\, d^3 r'\, \underbrace{f(r - r_a)}_{f_1(r)} \underbrace{f(r' - r_b)}_{f_2(r')} \, |r r'\rangle.$$

Prenons $\int_r (f_1(r))^2 = 1$ f réel.

$\langle \Psi | N(x) | \Psi \rangle$

calculs d'abord $\langle r_1 r_1' | N(x) | r_2 r_2' \rangle$.

$$= \int \frac{dq_1 dq_1'}{(2\pi)^3} \frac{dq_2 dq_2'}{(2\pi)^3} \frac{dk\, dk'}{(2\pi)^6\, 2\sqrt{\omega_k \omega_{k'}}} e^{i(q_1 r_1 + q_1' r_1' - q_2 r_2 - q_2' r_2' + (k' - k)x)}$$

$$\times \frac{1}{2(2\pi)^6\, 4\sqrt{\omega_{q_1}\omega_{q_1'}\omega_{q_2}\omega_{q_2'}}} \underbrace{\langle 0 | a_{q_1'} a_{q_1} a_k^+ a_{k'} a_{q_2}^+ a_{q_2'}^+ | 0 \rangle}_{I}$$

$$I = \langle 0 | a_{q_1'} a_{q_1} a_k^+ a_{q_2}^+ a_{k'} a_{q_2'}^+ | 0 \rangle + (2\pi)^3 2\omega_{k'} \delta(k' - q_2) \langle 0 | a_{q_1'} a_{q_1} a_k^+ a_{q_2'}^+ \ldots$$

$$= (2\pi)^3\, 2\omega_{k'} \left\{ \langle 0 | a_{q_1'} a_{q_1} a_k^+ a_{q_2}^+ | 0 \rangle \delta(k' - q_2') + \langle 0 | a_{q_1'} a_{q_1} a_k^+ a_{q_2'}^+ | 0 \rangle \delta(k' - q_2) \right\}.$$

(s 2 fois le "vrai" terme (modulo permutation de k et q_2')

$$\langle 0| a_{q_1'} a_{q_1} a^\dagger_k a^\dagger_{q_2}|0\rangle = \langle 0| a_{q_1'} a^\dagger_k a_{q_1} a^\dagger_{q_2}|0\rangle$$
$$+(2\pi)^3 2\omega_k \langle 0| a_{q_1'} a^\dagger_{q_2}|0\rangle \delta(k-q_1)$$
$$= \langle 0| a_{q_1'} a^\dagger_k |0\rangle (2\pi)^3 2\omega_{q_1}\delta(q_1-q_2) + (2\pi)^6 4\omega_k \omega_{q_1'} \delta(k-q_1)\delta(q_1'-q_2)$$
$$= (2\pi)^6 4\omega_k\omega_{q_1}\left\{ \delta(q_1-q_2)\delta(q_1'-k) + \delta(k-q_1)\delta(q_1'-q_2)\right\}.$$

$$I = (2\pi)^9 8\,\omega_k\omega_{k'}\omega_{q_1}\left\{\delta(k'-q_2')\left(\delta(q_1-q_2)\delta(q_1'-k) + \delta(k-q_1)\delta(q_1'-q_2)\right) + q_2' \leftrightarrow q_2\right\}.$$

$$\langle x_2| N(x)|x\rangle = \frac{1}{2(2\pi)^9}\int dq_1\, dq_1'\, dq_2\, dq_2'\, dk\, dk' \left\{ \delta(k'-q_2')\left(\delta(q_1-q_2)\delta(q_1'-k) + {}_{q_1\leftrightarrow q_1'}\right) + q_2' \leftrightarrow q_2\right\}.$$

$$e^{i(\qquad)}.$$

$$= \frac{1}{2(2\pi)^9}\int dk\, dk'\, dq_1 \left\{ e^{i\left(k'(x-x_2') + q_1(x_1-x_2) + k(x+x_1')\right)} + q_1 \leftrightarrow q_1' \right\} + q_2' \leftrightarrow q_2 \Big\}$$

$= \cancel{\delta(x-x_2')\delta(x_1-x_2)\delta(x_1'-x) + \delta(x-x_2')\delta(x_1'-x_2)\delta(x_1-x)}$
$\cancel{+\delta(x-x_2)\delta(x_1-x_2')\delta(x_1'-x) + \delta(x-x_2)\delta(x_1'-x_2')\delta(x_1-x)}$

$$= \frac{1}{2}\Big\{ \delta(r-r'_2)\,\delta(r_1-r_2)\,\delta(r'_1-r) + \delta(r-r'_2)\,\delta(r'_1-r_2)\,\delta(r_1-r)$$
$$+ \delta(r-r_2)\,\delta(r_1-r'_2)\,\delta(r'_1-r) + \delta(r-r_2)\,\delta(r'_1-r'_2)\,\delta(r_1-r) \Big\}.$$

$$\langle \psi | N(r) | \psi \rangle = \frac{1}{2}\Big\{ f_2^2(r) + f_1^2(r) + 2 f_1(r) f_2(r) \int f_1(r) f_2(r)\, d^3r$$

Normalisation $\langle \psi | \psi \rangle = \frac{1}{2}\Big\{ 1 + \Big(\int d^3r\, f_1(r)\Big)^2 \Big\}.$

$$\langle N(r) \rangle = \frac{f_2^2(r) + f_1^2(r) + 2 f_1(r) f_2(r)\,(f_1 f_2)}{1 + (f_1 f_2)^2}$$

σ a interférence entre les 2 particules.

$$f_1(r) = \frac{1}{(\pi\ell^2)^{3/4}}\, e^{-\frac{(r-r_a)^2}{2\ell^2}} \qquad f_2(r) = \frac{1}{(\pi\ell^2)^{3/4}}\, e^{-\frac{(r-r_b)^2}{2\ell^2}}$$

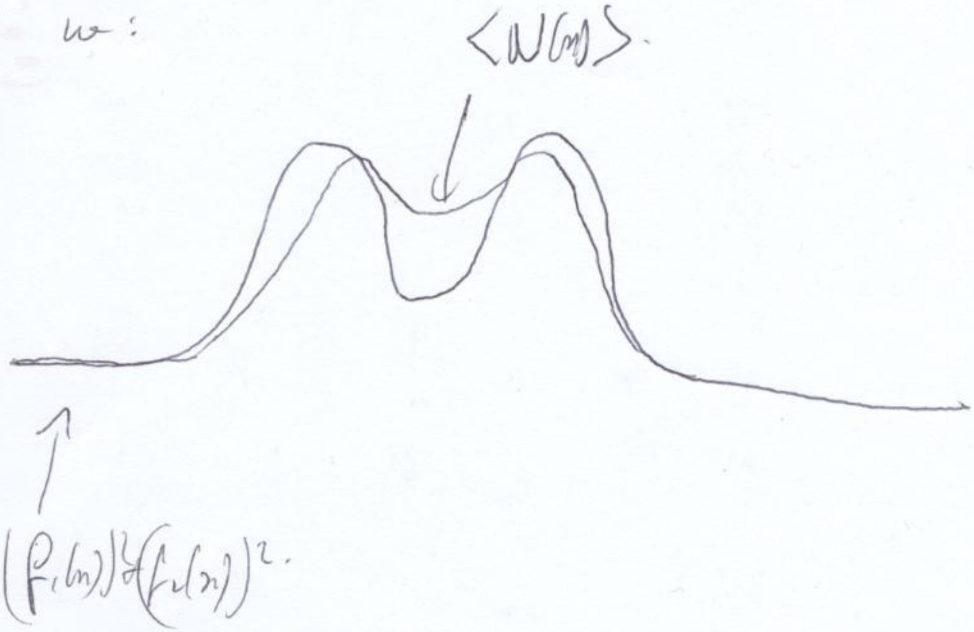
$$\int_r f_1(r) f_2(r) = \int_r \frac{1}{(\pi\ell^2)^{3/2}}\, e^{-\frac{1}{2\ell^2}\left\{ 2r^2 - 2r(r_a+r_b) + r_a^2 + r_b^2 \right\}}$$
$$= \int_r \frac{1}{(\pi\ell^2)^{3/2}}\, e^{-\frac{1}{\ell^2}\left\{ \left(r - \frac{r_a+r_b}{2}\right)^2 - \frac{(r_a+r_b)^2}{4} + \frac{r_a^2+r_b^2}{2} \right\}}$$
$$= e^{-\frac{(r_a-r_b)^2}{4\ell^2}}$$

Si $r_a = r_b \Rightarrow \int f_1 f_2 = 1$

$\langle N(x) \rangle = 2 f_1^2(x) \left(= f_1(x)^2 + f_2(x)^2 \right)$.

Si r_a et r_b très séparés: $\int f_1 f_2 = 0 \Rightarrow \langle N(x) \rangle = f_1^2(x) + f_2^2(x)$

Si non:



$\langle N(x) \rangle$ plus grand quand on a un recouvrement entre les 2 fonctions d'onde, plus petit si non...