Feuille 4

$\boxed{I}$

1) $(m, 0, 0, 0)$.

2) $p'^\mu = \left(\delta^\mu{}_\nu + \omega^\mu{}_\nu\right) p^\nu = \left(m, -m\vec{\phi}\right)$

$= p^\mu \Rightarrow \vec{\phi} = 0$

Petit groupe : rotations pures. $\Rightarrow SO(3)$.

représentations de dimension $(2s+1)$ (s = spin demi entier).

3) $p^\mu = (E, 0, 0, E)$.

$$p'^\mu = \left(E - E\phi_3, -E(\phi_1 + \theta_2), -E(\phi_2 - \theta_1), E - E\phi_3\right).$$

$= p^\mu \Rightarrow \phi_3 = 0$

$\phi_1 + \theta_2 = 0$

$\phi_2 - \theta_1 = 0$

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & \theta_2 & -\theta_1 & 0 \\ \theta_2 & 0 & \theta_3 & -\theta_2 \\ -\theta_1 & 0 & 0 & \theta_1 \\ 0 & \theta_2 & -\theta_1 & 0 \end{pmatrix} = i\,\theta_1 \overbrace{\left(J_1 + K_2\right)}^{\Pi_1} + i\,\theta_2 \overbrace{\left(J_2 - K_1\right)}^{\Pi_2}$$

$$+ i\,\theta_3\, J_3.$$

$$[J_3, \Pi_1] = [J_3, J_1] + [J_3, K_2]$$

$$= i\,(J_2 - K_1) = i\,\Pi_2$$

$$[J_3, \Pi_2] = [J_3, J_2] - [J_3, K_1]$$

$$= i\,(-J_1 - K_2) = -i\,\Pi_1$$

$$[\Pi_1, \Pi_2] = [J_1 + K_2, J_2 - K_1] = [J_1, J_2] + [K_2 J_2] - [J_1$$

$$- [K_2 K_1]$$

$$= i\,J_3 - iJ_3 = 0$$

$[J_3, \Pi_1] = i\,\Pi_2$ $\quad [J_3, \Pi_2] = -\,i\Pi_1$ $\quad [\Pi_1, \Pi_2] = 0.$

5) $$[J_3, \Pi_1^2] = J_3 \Pi_1^2 - \Pi_1^2 J_3 = \Pi_1 J_3 \Pi_1 + i\Pi_2 \Pi_1 - \Pi_1^2 J_3$$

$$= 2i\,\Pi_1 \Pi_2$$

$$[J_3, \Pi_2^2] = -2i\,\Pi_1 \Pi_1$$

$\Rightarrow [J_3, \Pi_1^2 + \Pi_2^2] = 0$ et $[\Pi_i, \Pi_1^2 + \Pi_2^2] = 0.$

$\hookrightarrow$ $\Pi_1^2 + \Pi_2^2$ opérateur de Casimir

6) les translations: $P_x\, P_y$ $\qquad [P_i P_j] = 0$

les rotations : J_z $\qquad [J_i, P_j] = i\,\varepsilon_{ijk} P_k.$

$\hookrightarrow [J_3, P_1] = i\,P_2$ $\quad [J_3, P_2] = -iP_1$ $\Rightarrow$ même relation de commutation...

2 1)
$$\partial_\mu(\Box A^\mu) = \Box \partial_\mu A^\mu = 0$$
$$= \partial_\mu J^\mu$$

↳ $\partial_\mu J^\mu = 0$ conservation de la charge

↳ $\left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \vec{j} = 0 \right.$

$\left. \int d^3x \, \frac{\partial \rho}{\partial t} + \vec{\nabla}\vec{j} = 0 \Rightarrow \frac{dQ}{dt} + \int_S \vec{j} \cdot d\vec{S} = 0 . \right)$

$A^\mu = B^\mu \, e^{i(\omega t - kz)}$ $\qquad \frac{\omega}{k} = c = 1 .$

$\partial_\mu A^\mu = \left(i\omega B^0 - ik B^3 \right) e^{i(\omega t - kz)}$.

↳ $B^0 - B^3 = 0$.

$$A^\mu = \begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix} e^{i(\omega t - kz)} = \begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^0 \end{pmatrix} e^{i\omega(t-z)}$$

En fait B^0 n'intervient pas des $\vec{E}$ et $\vec{B}$:

$E_i = -\partial_i A^0 - \partial_t A^i$
$E_3 = i\omega B^0 - i\omega B^0 = 0$.

$B_i = \vec{\nabla} \times \vec{A}$ ne dépend pas de B^0

En Fait $\begin{pmatrix} B^0 \\ 0 \\ 0 \\ B^0 \end{pmatrix} e^{i\omega(t-z)} = \partial^\mu \, B^0 e^{i\omega(t-z)} \Rightarrow$ par un changement de jauge, on peut éliminer B^0 ...

$$A^{*} = \left(A_+ \vec{e}_+ + A_- \vec{e}_-\right) e^{i\omega(t-z)}.$$

3) sous rotation. $\vec{e}_1{}' = \vec{e}_1 \cos\theta - \vec{e}_2 \sin\theta$
$\vec{e}_2{}' = \vec{e}_2 \cos\theta + \vec{e}_1 \sin\theta.$

(attention ! les vecteurs se transforment avec un signe opposé aux coordonnées

$$\vec{e}_\pm{}' = \vec{e}_1 \cos\theta - \vec{e}_2 \sin\theta \pm i\left(\vec{e}_2 \cos\theta + \vec{e}_1 \sin\theta\right)$$
$$= \cos\theta\left(\vec{e}_1 \pm i\vec{e}_2\right) + \sin\theta\left(-\vec{e}_2 \pm i\vec{e}_1\right)$$
$$= \cos\theta\left(\vec{e}_1 \pm i\vec{e}_2\right) \pm i\sin\theta\left(\vec{e}_1 \pm i\vec{e}_2\right)$$
$$= e^{\pm i\theta}\, \vec{e}_\pm$$

$$\hookrightarrow A' = \left(A_+ e^{i\theta} \vec{e}_+ + A_- e^{-i\theta} \vec{e}_-\right) e^{i\omega(t-z)}.$$

hélicités : ± 1

$h=+1$ correspond à $\vec{e}_+$
$h=-1$ " " $\vec{e}_-$ } polarisations circulaires.

pas d'hélicité $h=0$

4) $\partial_\alpha \Box \bar{h}^{\alpha\beta} = 0 \Rightarrow \partial_\alpha T^{\alpha\beta} = 0$ conservation de quelque chose (énergie et impulsion ...)

$\mathrm{Tr}\, \bar{h} = 0$

si $\bar{h}^{\mu\nu} = \underbrace{\ell^{\mu\nu}}_{\text{constante}} e^{i\omega(t-z)}.$

$$\partial_\mu \bar{h}^{\mu\nu} = 0 \quad \Rightarrow \quad i\omega\left(\bar{h}^{0\nu} - \bar{h}^{3\nu}\right) = 0$$

en fait $\ell_{\mu\nu} = A_+ M_+ + A_\times M_\times$

$$\hookrightarrow \bar{h}^{\mu\nu} = \left(A_+ M_+ + A_\times M_\times\right) e^{i\omega(t-z)}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(t-z)}$$

Rotation: $R: \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$\begin{pmatrix} A_+ & A_\times \\ A_\times & -A_+ \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$${}^tR \; M \; R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_+c + A_\times s & A_\times c - A_+ s \\ -A_+ s + A_\times c & -A_\times s - A_+ c \end{pmatrix}$$

$$= \begin{pmatrix} A_+(c^2-s^2) + 2A_\times cs & A_+(-2sc) + A_\times(c^2-s^2) \\ A_+(-2sc) + A_\times(c^2-s^2) & A_+(c^2-s^2) - 2A_\times cs \end{pmatrix}$$

$$A_+' = A_+ \cos 2\theta + A_\times \sin 2\theta$$

$$A_\times' = A_\times \cos 2\theta - A_+ \sin 2\theta$$

$$e_\pm = A_+ \pm i A_\times$$

$$e_\pm' = e^{\pm 2i\theta} e_\pm$$

is spin 2

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1) Schrödinger: $i\partial_t \psi = \left(-\frac{\Delta}{2m} + V\right)\psi.$

$$\partial_t \psi = \left(i\frac{\Delta}{2m} - iV\right)\psi.$$

$$\psi^* \partial_t \psi = \frac{i}{2m}\psi^* \Delta \psi - iV\psi^*\psi$$

$$\underline{\psi\partial_t \psi^* = -\frac{i}{2m}\psi\Delta\psi^* + iV\psi^*\psi.}$$

$$\partial_t |\psi|^2 = \frac{i}{2m}\left(\psi^*\Delta\psi - \psi\Delta\psi^*\right)$$

$$\vec{J} = -\frac{i}{2m}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right).$$

2) $\partial_t \rho = \frac{i}{2m}\left(\psi^*\partial_t^2\psi - \psi\partial_t^2\psi^*\right)$

$$\nabla\vec{J} = -\frac{i}{2m}\left(\psi^*\Delta\psi - \psi\Delta\psi^*\right).$$

$$\partial_t\rho + \vec{\nabla}\vec{J} = \frac{i}{2m}\left[\psi^*\Box\psi - \psi\Box\psi^*\right]$$

$$\left[-m^2\psi^*\psi + m^2\psi^*\psi\right] = 0. \quad \underline{\underline{OK}}$$

3) $i\partial_t(i\partial_t\psi) = i\partial_t H\psi = H(i\partial_t\psi) = H^2\psi.$

$$-\partial_t^2\psi = H^2\psi.$$

$$H^2 = \beta^2 m^2 - \alpha_i \alpha_i \partial_i \partial_i - i m \partial_i (\alpha_i \beta + \beta \alpha_i)$$

$$- (\alpha_i \alpha_j + \alpha_j \alpha_i) \partial_i \partial_j$$

on doit avoir $\beta^2 = 1$

$$\alpha_i \alpha_i = 1 \quad \alpha_i \alpha_j + \alpha_j \alpha_i = \{\alpha_i, \alpha_j\} = 0$$

$$\alpha_i \beta + \beta \alpha_i = \{\alpha_i, \beta\} = 0$$

4]

$$\{\gamma^0, \gamma^0\} = \{\beta, \beta\} = 2.$$

$$\{\gamma^0, \gamma^i\} = \{\beta, \beta\alpha^i\} = \beta^2 \alpha^i + \beta \alpha^i \beta$$

$$= \beta^2 \alpha^i - \beta^2 \alpha^i = 0.$$

$$\{\gamma^i, \gamma^j\} = \beta \alpha^i \beta \alpha^j + \beta \alpha^j \beta \alpha^i$$

$$= -\beta^2 (\alpha^i \alpha^j + \alpha^j \alpha^i) = -\{\alpha^i, \alpha^j\} = -2$$

$\hookrightarrow \{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}.$

$$\beta \times (\ \ ) \; i \partial_t \Psi = -i \vec{\alpha} \vec{\nabla} \Psi + \beta m) \Psi.$$

$$i \beta \partial_t \Psi = (-i \beta \vec{\alpha} \vec{\nabla} \beta + m) \Psi$$

$$i \gamma^\mu \partial_\mu \Psi = m \Psi \Rightarrow (i \not{\partial} - m) \Psi = 0$$

$\boxed{4}$

$$\left(i\gamma^\mu \partial'_\mu - m\right)\psi(x') = 0$$

$$\left(i\gamma^\mu \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu - m\right) S(\Lambda)\psi(x) = 0$$

$$\left(i(\Lambda^{-1})^\nu{}_\mu\, \gamma^\mu \partial_\nu - m\right) S(\Lambda)\psi(x) = 0$$

multiplie à G par $S^{-1}(\Lambda)$.

$$\left(i\,\Lambda^{-1\nu}{}_\mu\, S^{-1}(\Lambda)\gamma^\mu S(\Lambda)\,\partial_\nu - m\right)\psi = 0$$

$$\hookrightarrow\quad \Lambda^{-1\nu}{}_\mu\, S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \gamma^\nu$$

$$S^{-1}(\Lambda)\,\gamma^\rho S(\Lambda) = \Lambda^\rho{}_\nu\,\gamma^\nu$$

$$\left(\mathbb{1} + \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)\gamma^\rho\left(\mathbb{1} - \frac{i}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right) = \left(\eta^\rho{}_\nu + \omega^\rho{}_\nu\right)$$

$$\gamma^\rho + \frac{i}{4}\omega^{\mu\nu}\left[\sigma_{\mu\nu}, \gamma^\rho\right] = \gamma^\rho + \omega^\rho{}_\nu\,\gamma^\nu$$
$$+\ \omega^{\mu\nu}\ \underbrace{\gamma_\nu \eta^\rho{}_\mu}_{\text{non antisymétrique}}$$

$$\cancel{\hookrightarrow \frac{i}{4}\left[\sigma_{\mu\nu}, \gamma^\rho\right] = \gamma_\nu \eta^\rho{}_\mu}$$

$$\frac{i}{4}\omega^{\mu\nu}\left[\sigma_{\mu\nu}, \gamma^\rho\right] = \frac{\omega^{\mu\nu}}{2}\left(\gamma_\nu \eta^\rho{}_\mu - \gamma_\mu \eta^\rho{}_\nu\right).$$

$$\hookrightarrow\quad \left[\sigma_{\mu\nu}, \gamma^\rho\right] = -2i\left(\eta^\rho{}_\mu \gamma_\nu - \eta^\rho{}_\nu \gamma_\mu\right).$$

$$\sigma_{\mu\nu} = \frac{i}{2}\left(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu\right).$$

$$\left[\sigma_{\mu\nu}, \gamma^\rho\right] = \frac{i}{2}\left(\gamma_\mu \gamma_\nu \gamma^\rho - \gamma_\nu \gamma_\mu \gamma^\rho - \gamma^\rho \gamma_\mu \gamma_\nu + \gamma^\rho \gamma_\nu \gamma_\mu\right.$$

$$= \frac{i}{2}\left(-\gamma_\mu \gamma^\rho \gamma_\nu + 2\gamma_\mu \eta^\rho{}_\nu + \gamma_\nu \gamma^\rho \gamma_\mu - 2\gamma_\nu \eta_\mu{}^\rho \right.$$
$$\left. - \gamma^\rho \gamma_\mu \gamma_\nu + \gamma^\rho \gamma_\nu \gamma_\mu\right).$$

$$= \frac{i}{2}\left(-2\eta^\rho{}_\mu \gamma_\nu + 2\gamma_\mu \eta^\rho{}_\nu - 2\eta^\rho{}_\nu \gamma_\mu - 2\gamma_\nu \eta^\rho_\mu\right.$$

$$= 2i\left(\gamma_\mu \eta^\rho{}_\nu - \gamma_\nu \eta^\rho{}_\mu\right). \qquad \text{OK.}$$

$$\sigma_{0i} = \frac{i}{2}\left[\gamma_0, \gamma_i\right]$$

$$= \frac{i}{2}\left\{-\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$$

$$= \frac{i}{2}\left\{-\begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} + \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}\right\}$$

$$= i\begin{pmatrix} +\sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}.$$

$$\sigma_{ij} = \frac{i}{2}\left[\gamma_i, \gamma_j\right] = \frac{i}{2}\left\{\begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix} - (i \leftrightarrow j)\right.$$

$$= \frac{i}{2}\left\{\begin{pmatrix} -\sigma_i\sigma_j & 0 \\ 0 & -\sigma_i\sigma_j \end{pmatrix} - \begin{pmatrix} -\sigma_j\sigma_i & 0 \\ 0 & -\sigma_j\sigma_i \end{pmatrix}\right\}.$$

$$= \varepsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}.$$

$\boxed{5}$

1] Dirac: $(i\cancel{\partial} - m)\psi = 0$

avec champ electromagnétique: $(i\cancel{\partial} - e\cancel{A} - m)\psi = 0$

$$\hookrightarrow \left(i\partial_0 + i\partial_i\gamma^0\gamma^i - e\gamma^0\cancel{A} - m\gamma^0\right)\psi = 0$$

$$\hookrightarrow i\partial_0\psi = \left(-i\gamma^0\gamma^i\partial_i + e\gamma^0\cancel{A} + m\gamma^0\right)\psi$$

2] $$\gamma^0\gamma^i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}.$$

$$i\partial_t\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \begin{pmatrix} eA^0 + m & -(i\partial_i + A^i)\sigma_i \\ -(i\partial_i + A^i)\sigma_i & eA^0 - m \end{pmatrix}\begin{pmatrix} \varphi \\ \chi \end{pmatrix}.$$

$$i\partial_t\varphi = (eA^0 + m)\varphi - (i\partial_i + A^i)\sigma_i\chi.$$

$$i\partial_t\chi = (eA^0 - m)\chi - (i\partial_i + A^i)\sigma_i\varphi.$$

3] $$\varphi = e^{-imt}\,\Phi$$

$$\chi = e^{-imt}\,X.$$

$$\hookrightarrow \begin{cases} i\partial_t\Phi = (eA^0\ \ )\,\Phi - (i\partial_i + A^i)\sigma_i X \\ i\partial_t X = (eA^0 - 2m)\,X - (i\partial_i + A^i)\sigma_i\Phi \end{cases}$$

X varie lentement $\Rightarrow \partial_t \cancel{X} = O(m^0)$.

eA^0 petit $\left(O(m^0)\right)$.

$\hookrightarrow X = -\frac{(i\partial_i + A^i)}{2m}\sigma_i \Phi$

$$i\partial_t \phi = eA^0\phi + \frac{1}{2m}\left[(i\partial_i + eA^i)\sigma_i\right]^2 \Phi$$

$$\underbrace{(i\partial_i + eA^i)(i\partial_j + eA^j)\left(\delta_{ij} + i\varepsilon_{ijk}\sigma_k\right)}_{M}\phi$$

$$M = (i\partial_i + eA^i)^2 - e\varepsilon_{ijk}\sigma_k\left(\partial_i A^j + A^i\partial_j\right)$$

$$= (i\partial_i + eA^i)^2 - e\varepsilon_{ijk}\sigma_k\left(\partial_i A^j\right)$$

$$= (i\partial_i + eA^i)^2 - eB_k\sigma_k$$

$$\hookrightarrow i\partial_t\Phi = eA^0\phi + \frac{(i\partial_i + eA^i)^2}{2m}\phi \underbrace{- \frac{e\,\vec{B}\vec{\sigma}}{2m}\phi}_{\text{couplage spin / champ magnétique}}$$