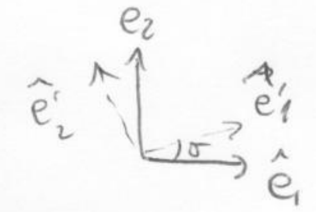
Feuille 2

1

1)



$$x' = \cos\theta\, x + \sin\theta\, y$$
$$y' = -\sin\theta\, x + \cos\theta\, y$$
$$z' = z$$

$$R_{ij} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2) R_{ij} peut dépendre de tenseurs construits à partir de n_i, δ_{ij}, ε_{ijk}.

Pour faire un tenseur à 2 indices (de rang 2) :

$n_i n_j$, δ_{ij}, $\varepsilon_{ijk} n_k$.

$$R_{ij} = A\delta_{ij} + B n_i n_j + C \varepsilon_{ijk} n_k.$$

3) Pour une rotation autour de $\hat{e}_3$

$$R_{ij} = \begin{pmatrix} A & C & 0 \\ -C & A & 0 \\ 0 & 0 & A+B \end{pmatrix}$$

Par identification : $A = \cos\theta$

$B = 1 - \cos\theta$

$C = \sin\theta$

↳ Pour une rotation quelconque :

$$R_{ij} = \cos\theta\, \delta_{ij} + (1-\cos\theta) n_i n_j + \sin\theta\, \varepsilon_{ijk} n_k$$

$\boxed{2}$

1) $[J_i, J_j]_{kl} = (J_i)_{km}(J_j)_{ml} - (J_j)_{km}(J_i)_{ml}$

$$= -\varepsilon_{kmi}\,\varepsilon_{mlj} + (i \leftrightarrow j)$$

$$= -(\delta_{il}\delta_{jk} - \cancel{\delta_{ij}\delta_{kl}}) + (\delta_{jl}\delta_{ik} - \cancel{\delta_{ij}}\delta_{kl})$$

$$= \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$$

$$= \varepsilon_{ijm}\,\varepsilon_{klm}$$

$$= i\,\varepsilon_{ijm}\,(-i\,\varepsilon_{klm})$$

$$[J_i, J_j] = i\,\varepsilon_{ijk}\,J_k.$$

2) $(J_a n_a)^2_{ij} = (J_a J_b)_{ij}\, n_a n_b$

$$= -\,\varepsilon_{ika}\,\varepsilon_{kjb}\, n_a n_b$$

$$= (\delta_{ij}\delta_{ab} - \delta_{ib}\delta_{ja})\, n_a n_b$$

$$= \delta_{ij} - n_i n_j$$

$$(J_a n_a)^3_{ij} = (\delta_{ik} - n_i n_k)(J_a)_{kj}\, n_a$$

$$= -i(\delta_{ik} - n_i n_k)\,\varepsilon_{kja}\, n_a$$

$$= -i\,\varepsilon_{ija}\, n_a$$

$$(J_a n_a)^3 = J_a n_a$$

$$\left(J_a n_a\right)^{2\alpha}_{ij} = \delta_{ij} - n_i n_j \qquad \forall \alpha > 0$$

$$\left(J_a n_a\right)^{2\alpha+1}_{ij} = \left(J_a n_a\right)_{ij} \qquad \forall \alpha \geq 0$$

3) $$\left[\exp(i\theta\, n_a J_a)\right]_{ij} = \sum_\alpha \frac{1}{\alpha!}\left(i\theta\, n_a J_a\right)^\alpha$$

$$= \delta_{ij} + \sum_{\alpha=1}^{\infty} \frac{(-1)^\alpha}{(2\alpha)!}\,\theta^{2\alpha}\left(\delta_{ij} - n_i n_j\right) + \sum_{\alpha=0}^{\infty} i\,\frac{(-1)^\alpha}{(2\alpha+1)!}\left(n_a J_a\right)_{ij}$$

$$= \delta_{ij} + (\cos\theta - 1)(\delta_{ij} - n_i n_j) + i\sin\theta\,(n_a J_a)_{ij}$$

$$= \cos\theta\,\delta_{ij} + (1-\cos\theta)\, n_i n_j + \sin\theta\,\varepsilon_{ija} n_a .$$

correspond à l'expression précédente...

$\boxed{3}$

1) $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbb{1}$.

$$\sigma_1\sigma_2 = \begin{pmatrix} i & \\ & -i \end{pmatrix} = i\,\sigma_3$$

$$\sigma_2\sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\,\sigma_1$$

$$\sigma_3\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\,\sigma_2 .$$

$$(\sigma_i\sigma_j)^\dagger = \sigma_j^\dagger\sigma_i^\dagger = \sigma_j\sigma_i$$

$\hookrightarrow$ $\sigma_2\sigma_1 = -i\,\sigma_3$

$\sigma_3\sigma_2 = -i\,\sigma_1$

$\sigma_1\sigma_3 = -i\,\sigma_2$

$\hookrightarrow$ $\sigma_i\sigma_j = \delta_{ij}\,\mathbb{1} + i\,\varepsilon_{ijk}\,\sigma_k$.

2) $$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = \frac{\delta_{ij}}{4}\mathbb{1} + \frac{i}{4}\varepsilon_{ijk}\sigma_k - \frac{\delta_{ij}}{4} - \frac{i}{4}\varepsilon_{jik}\sigma_k = i\,\varepsilon_{ijk}\,\frac{\sigma_k}{2}$$

3) $$(n_i\sigma_i)^2 = n_i n_j\left(\delta_{ij}\mathbb{1} + i\,\varepsilon_{ijk}\sigma_k\right) = \mathbb{1}$$

$(n_i\sigma_i)^{2\alpha} = \mathbb{1} \qquad \forall \alpha \geq 0$

$(n_i\sigma_i)^{2\alpha+1} = n_i\sigma_i \qquad \forall \alpha \geq 0.$

4] $\exp\left(i\,\theta\, n_i \frac{\sigma_i}{2}\right) = \sum_{\alpha=0}^{\infty} (-1)^\alpha \left(\frac{\theta}{2}\right)^{2\alpha} \mathbb{1} + \sum_{\alpha=0}^{\infty} i\,(-1)^\alpha \left(\frac{\theta}{2}\right)^{2\alpha+1} n_i\sigma_i$

$= \cos\frac{\theta}{2}\,\mathbb{1} + i \sin\frac{\theta}{2}\; n_i\sigma_i$

$\boxed{4}$ 1] $\frac{1}{2} \mathrm{Tr}\,(\cancel{U}\cancel{V}) = \frac{1}{2}\, U_i V_j\, \mathrm{Tr}(\sigma_i\sigma_j).$

$= \frac{1}{2} U_i V_j\, \mathrm{Tr}\left(\delta_{ij}\mathbb{1} + i\,\varepsilon_{ijk}\sigma_k\right).$

$= U_i V_i = \vec{U}.\vec{V}$

$-\frac{i}{2}\mathrm{Tr}(\cancel{U}\cancel{V}\cancel{W}) = -\frac{i}{2}\, U_i V_j W_k\, (\sigma_i\sigma_j\sigma_k).$

$= -\frac{i}{2} U_i V_j W_k \left((\delta_{ij}\mathbb{1} + i\,\varepsilon_{ijl}\sigma_l)\, \sigma_k \right).$

$= -\frac{i}{2} U_i V_j W_k \left(\delta_{ij}\sigma_k + i\,\varepsilon_{ijk}\mathbb{1} - \varepsilon_{ijl}\varepsilon_{lkm}\sigma_m \right.$

$= \cancel{\tfrac{1}{2}}\, U_i V_j W_k\, \varepsilon_{ijk} = (\vec{U}\times\vec{V}).\vec{W}$

2] $\mathrm{Tr}(\cancel{e}'_1\cancel{e}'_2\cancel{e}'_3) = \mathrm{Tr}\left(a^+ \cancel{e}_1\, a\; a^+\cancel{e}_2\, a\; a^+\cancel{e}_3\, a\right).$

$= \mathrm{Tr}\,(\cancel{e}_1\cancel{e}_2\cancel{e}_3)$

↳ $(\vec{e}_1\times\vec{e}_2).\vec{e}_3 = (\vec{e}\,'_1\times\vec{e}\,'_2).\vec{e}\,'_3$

↳ la transformation change une ~~opère~~ base directe en une base directe

⇒ ce n'est pas une symétrie par rapport à un plan ⇒ ça correspond à une rotation.

5] point de vue passif : le vecteur ne change pas :

$$\vec{U} = U_j \vec{e}_j = U'_j \vec{e}'_j$$

produit scalaire $\vec{e}'_i$

$$U_j \vec{e}_j \cdot \vec{e}'_i = U'_j \vec{e}'_j \vec{e}'_i = U'_i$$

$$\hookrightarrow U'_i = \underbrace{\vec{e}'_i \cdot \vec{e}_j}_{R_{ij}} U_j$$

6] $$R_{ij} = \vec{e}'_i \vec{e}_j = \frac{1}{2} \mathrm{Tr}\, e'_i e_j = \frac{1}{2} \mathrm{Tr}\, a^+ e_i a\, e_j$$

$$= \frac{1}{2} \mathrm{Tr}\left(a^+ \sigma_i a \sigma_j\right)$$

$$a^+ \sigma_i = \left(\cos\frac{\theta}{2} \mathbb{1} - i \sin\frac{\theta}{2}\, n_a \sigma_a\right) \sigma_i$$

$$= \cos\frac{\theta}{2} \sigma_i - i \sin\frac{\theta}{2} \left(n_i \mathbb{1} + i n_a \varepsilon_{aik} \sigma_k\right)$$

$a\sigma_j$: changer $i \leftrightarrow j$
$\theta \leftrightarrow -\theta$.

$$a \sigma_j = \cos\frac{\theta}{2} \sigma_j + i \sin\frac{\theta}{2} \left(n_j \mathbb{1} + i n_k \varepsilon_{kjl} \sigma_l\right)$$

$$a^+ \sigma_i a \sigma_j = \cos\frac{\theta}{2}^2 \delta_{ij} + i \sin\frac{\theta}{2} \cos\frac{\theta}{2} \left(i n_k \varepsilon_{kji} - i n_a \varepsilon_{aij}\right) + \sin^2\frac{\theta}{2}\left(n_i n_j - n_a n_k \underbrace{\varepsilon_{aik}\, \varepsilon_{kjk}}_{(\delta_{ak}\delta_{ij} - \delta_{aj}\delta_{ki})}\right)$$

$$= \cos^2\frac{\theta}{2} \delta_{ij} + 2 \sin\frac{\theta}{2} \cos\frac{\theta}{2}\, \varepsilon_{ijk} n_k + \sin^2\frac{\theta}{2} \left(2 n_i n_j - \delta_{ij}\right)$$

$$= \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)\delta_{ij} + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,\varepsilon_{ijk}n_k + 2\,n_in_j\sin^2\frac{\theta}{2}.$$

$$R_{ij} = \cos\theta\,\delta_{ij} + (1-\cos\theta)\,n_in_j + \sin\theta\,\varepsilon_{ijk}n_k.$$

On retrouve toujours la même formule.

8] $S_0^2 + S_1^2 + S_2^2 + S_3^2 = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\,\hat{n}^2 = 1.$

$$R_{ij}^{(\underline{S})} = \left(S_0^2 - S_\alpha^2\right)\delta_{ij} + 2\,S_0\,S_k\,\varepsilon_{ijk} + 2\,S_i\,S_j$$

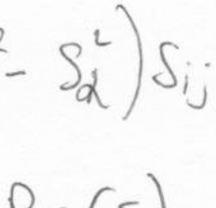
$R_{ij}(-\underline{S}) = R_{ij}(\underline{S})$: les points diamétralement opposés sont identifi

$$a_{ij}(\underline{S}) = S_0\,\mathbb{1} + i\,S_i\,\sigma_i$$

$$a_{ij}(-\underline{S}) = -\,a_{ij}(\underline{S}).$$

9]

S_2



RP_2

Si on combine 2 chemins

$\approx$

$\approx$

$\approx$

chemin trivial.

$\Rightarrow 1+1=0 \iff \mathbb{Z}_2$.

S_2 : tout chemin fermé se reforme sur le chemin trivial.

10) à tout chemin fermé, on peut associer un nombre algébrique : le nombre de tours effectués dans le sens trigo. On peut "sommer" les chemins $\Rightarrow$ on somme la variable comptant le nombre de tours $\Rightarrow$

$\mathbb{Z}$.

$\boxed{5}$ $[r_i, p_j] = i\hbar\delta_{ij}$

$$[\ell_i, r_j] = \varepsilon_{ikl}[r_k p_l, r_j] = \varepsilon_{ikl}(r_k p_l r_j - r_j r_k p_l)$$
$$= \varepsilon_{ikl}(r_k p_l r_j - r_k(p_l r_j + i\hbar\delta_{lj}))$$
$$= -i\hbar\,\varepsilon_{ikj} r_k = i\hbar\,\varepsilon_{ijk} r_k.$$

$$[\ell_i, p_j] = \varepsilon_{ikl}[r_k p_l, p_j] = \varepsilon_{ikl}(r_k p_l p_j - p_j r_k p_l)$$
$$= \varepsilon_{ikl}((p_j r_k + i\hbar\delta_{jk}) p_l - p_j r_k p_l)$$
$$= i\hbar\,\varepsilon_{ijk} p_k$$

$$[\ell_i, \ell_j] = \varepsilon_{jkl}[\ell_i, r_k p_l] = \varepsilon_{jkl}(\ell_i r_k p_l - r_k p_l \ell_i)$$
$$= \varepsilon_{jkl}(\ell_i r_k p_l - r_k \ell_i p_l + r_k \ell_i p_l - r_k p_l \ell_i)$$
$$= \varepsilon_{jkl}([\ell_i, r_k] p_l + r_k [\ell_i, p_l])$$
$$= i\hbar\,\varepsilon_{jkl}(\varepsilon_{ikm} r_m p_l + \varepsilon_{ilm} r_k p_m)$$
$$= i\hbar((\delta_{lm}\delta_{ij} - \delta_{li}\delta_{mj}) r_m p_l + (\delta_{ik}\delta_{jm} - \delta_{ij}\delta_{mk}) r_k p_m)$$
$$= i\hbar(r_i p_j - r_j p_i)$$
$$= i\hbar\,\varepsilon_{ijk}\ell_k$$

$$[\ell_i, A_j^2] = \ell_i A_j^2 - A_j \ell_i A_j + A_j \ell_i A_j - A_j^2 \ell_i$$
$$= [\ell_i, A_j] A_j + A_j [\ell_i, A_j]$$
$$= i\hbar(\varepsilon_{ijk} A_k A_j + \varepsilon_{ijk} A_j A_k) = 0.$$

$[l_i, f(A^2)] = 0$.

Ces résultats nous montrent, comme on s'y attend que l_i est le générateur des rotations (tous les vecteurs se transforment de la même façon, les scalaires ne se transforment pas).

↳ $[l_i, H] = 0$.

↳ les valeurs propres de L^2 et L_z sont de bons nombres quantiques pour décrire un état propre de H.

2) $$\dot{\vec{R}} = \vec{a} \times \vec{l} + \vec{v} \times \underbrace{\dot{\vec{l}}}_{=0 \text{ (potentiel central)}} - e^2 \frac{\vec{v}}{r} + e^2 \frac{\vec{r}}{r^2} \frac{dr}{dt}$$

$$r = \sqrt{r_i^2}$$

$$\frac{dr}{dt} = \frac{\vec{r} \cdot \vec{v}}{r}$$

$$\vec{a} = \frac{F}{m} = -\frac{e^2}{m} \frac{\vec{r}}{r^3}$$

$$\dot{\vec{R}} = -\frac{e^2}{r^3} \vec{r} \times (\vec{r} \times \vec{v}) - e^2 \frac{\vec{v}}{r} + e^2 \frac{\vec{r}}{r^3} (\vec{r} \cdot \vec{v})$$

$$= -\frac{e^2}{r^3} \vec{r} (\vec{r} \cdot \vec{v}) + e^2 \frac{\vec{v}}{r} - e^2 \frac{\vec{v}}{r} + e^2 \frac{\vec{r}}{r^3} (\vec{r} \cdot \vec{v})$$

3) $$= \vec{0}.$$

↳ voir page suivante...

4) $$R_i = \frac{1}{2m} \varepsilon_{ijk} (p_j l_k - l_j p_k) - e^2 \frac{\hat{r}_i}{\hat{r}}$$

$$= \frac{1}{2m} \varepsilon_{ijk} (p_j l_k - p_k l_j - i\hbar \varepsilon_{jkl} p_l) - e^2 \frac{r_i}{r}$$

$$= \frac{\varepsilon_{ijk} p_j l_k}{m} - \frac{i\hbar}{m} p_i - e^2 \frac{r_i}{r}.$$

3) $R_i^{*} = \frac{1}{2m}\varepsilon_{ijk}\left(p_j \ell_k - \ell_j p_k \right) - e^2 \frac{r_i}{r}$

$$R_i^{\dagger} = \frac{1}{2m}\varepsilon_{ijk}\left(\ell_k^{\dagger} p_j^{\dagger} - p_k^{\dagger} \ell_j^{\dagger} \right) - e^2 \frac{r_i^{\dagger}}{r^{\dagger}}$$

$$= \frac{1}{2m}\varepsilon_{ijk}\left(\ell_k p_j - p_k \ell_j \right) - e^2 \frac{r_i}{r}$$

$$= + \frac{1}{2m}\varepsilon_{ikj}\left(p_k \ell_j - \ell_k p_j \right) - e^2 \frac{r_i}{r}$$

$$= R_i$$

$$\ell_i R_i = \frac{\varepsilon_{ijk}}{m}\, \ell_i p_j \ell_k - \frac{i\hbar}{m}\, \ell_i p_i - \frac{e^2}{r}\, \ell_i r_i$$

$$= \frac{\varepsilon_{ijk}}{m}\left(p_j \ell_i \ell_k + i\hbar\, \varepsilon_{ijl}\, p_l \ell_k \right) - \frac{i\hbar}{m}\, \ell_i p_i - \frac{e^2}{r}\, \ell_i r_i$$

$$\varepsilon_{ijk}\, \ell_i \ell_j = 0$$

$$\ell_i p_i = \varepsilon_{ijk}\, r_j p_k p_i = 0$$

$$p_i \ell_i = \ell_i p_i + \underbrace{[p_i, \ell_i]}_{=0}$$

$$\ell_i r_i = r_i \ell_i = r_i\, \varepsilon_{ijk}\, r_j p_k = 0$$

$$\Rightarrow \vec{\ell}\cdot\vec{R} = 0.$$

5]
$$[R_i, H] = \Big[\overset{A}{\frac{\varepsilon_{ijk}}{m} p_j \ell_k} \overset{B}{- \frac{i\hbar}{m} p_i} \overset{C}{- e^2 \frac{r_i}{r}} , \overset{D}{\frac{p^2}{2m}} \overset{E}{- \frac{e^2}{r}} \Big]$$

$$[A, D] = \frac{\varepsilon_{ijk}}{2m^2}\left(\underbrace{p_j \ell_k p^2 - p_j p^2 \ell_k}_{p_j [\ell_k, p^2] = 0} + \underbrace{p_j p^2 \ell_k - p^2 p_j \ell_k}_{[p_j, p^2]\ell_k = 0} \right)$$

$$[A, D] = 0$$

$$[A, E] = -\frac{e^2}{m}\, \varepsilon_{ijk}\left(\underbrace{p_j \ell_k \frac{1}{r} - p_j \frac{1}{r} \ell_k}_{p_j \underbrace{[\ell_k, \frac{1}{r}]}_{=0}} + \underbrace{p_j \frac{1}{r} \ell_k - \frac{1}{r} p_j \ell_k}_{[p_j, \frac{1}{r}]\ell_k} \right.$$

$$[p_i, r_j^2] = p_i r_j^2 - r_j p_i r_j + r_j p_i r_j - r_j^2 p_i = \underbrace{[p_i, r_j]}_{=-i\hbar\,\delta_{ij}}\, r_j + r_j [p_i, r_j] = -2 i\hbar\, r_i$$

$$[p_i, f(r^2)] = -2i\hbar\, r_i\, f'(r^2).$$

en particulier:

$$\left[p_i, \frac{1}{r}\right] = -2i\hbar\, r_i \left(-\frac{1}{2}\right)\frac{1}{r^3}$$

$$= i\hbar \frac{r_i}{r^3}$$

$$\left[p_i, \frac{1}{r^3}\right] = 3i\hbar \frac{r_i}{r^5}$$

$\vdots$

$$[A, E] = -\frac{e^2}{m}\, i\hbar \frac{1}{r^3}\varepsilon_{ijk}\, r_j\, \ell_k$$

$$= -\frac{i\hbar e^2}{m}\frac{1}{r^3}\underbrace{\varepsilon_{ijk}\,\varepsilon_{k\ell m}\, r_j\, r_\ell\, p_m}_{-r^2 p_i + r_i(\vec{r}.\vec{p})}$$

$$[A, E] = +\frac{ie^2\hbar}{m}\left(\frac{1}{r}p_i - \frac{r_i}{r^3}(\vec{r}.\vec{p})\right)$$

$$[B, D] = 0$$

$$[B, E] = \frac{i\hbar e^2}{m}\left(p_i\frac{1}{r} - \frac{1}{r}p_i\right)$$

$$= \frac{i\hbar e^2}{m}\left(\frac{1}{r}p_i + i\hbar\frac{r_i}{r^3} - \frac{1}{r}p_i\right)$$

$$[B, E] = -\frac{\hbar^2 e^2}{m}\frac{r_i}{r^3}$$

$$[C, D] = -\frac{e^2}{2m}\left(\frac{1}{r}r_i p^2 - p^2 r_i \frac{1}{r}\right)$$

$$[P^2, r_i] = P_j P_j r_i - r_i P_j P_j$$
$$= P_j r_i P_j - i\hbar P_i - r_i P_j^2$$
$$= r_i P_j^2 - i\hbar P_i - r_i P_j^2$$
$$= -2i\hbar P_i$$

$$[C,D] = -\frac{e^2}{2m}\left(\frac{1}{r} r_i P^2 - r_i P^2 \frac{1}{r} + 2i\hbar P_i \frac{1}{r}\right)$$

$$[P^2, \cancel{\hbar}\tfrac{1}{r}] = \left(P_i P_i \frac{1}{r} - \frac{1}{r} P_i P_i\right)$$
$$= \left(P_i \frac{1}{r} P_i + i\hbar P_i r_i \frac{1}{r^3} - \frac{1}{r} P_i P_i\right)$$

Some

$\delta_{ii} =$

$=$

$$= i\hbar \frac{r_i}{r^3} P_i + i\hbar r_i P_i \frac{1}{r^3} + i\hbar(-i\hbar \times 3)\frac{1}{r^3}$$

$$= 2i\hbar \frac{r_i}{r^3} P_i + i\hbar r_i \left(3i\hbar \frac{r_i}{r^5}\right) + 3\hbar^2 \frac{1}{r^3}$$
$$= 2i\hbar \frac{1}{r^3} \vec{r}\cdot\vec{p}$$

$$[C,D] = -\frac{e^2}{2m}\left(\frac{1}{r} r_i P^2 - r_i \frac{1}{r} P^2 - r_i\left(2i\hbar \frac{1}{r^3} \vec{r}\cdot\vec{p}\right)\right.$$
$$\left. + 2i\hbar \frac{1}{r} P_i + 2i\hbar\left(i\hbar \frac{r_i}{r^3}\right)\right)$$

$$[C,D] = +\underline{\frac{e^2\hbar}{m}\left(i \frac{r_i}{r^3} \vec{r}\cdot\vec{p} - i \frac{1}{r} P_i\right.} \left. + \hbar \frac{r_i}{r^3}\right)$$

$$\underline{[C,E] = 0}$$

$$[R_i, H] = \frac{e^2\hbar}{m}\left(i\frac{4}{r}p_i - \cancel{i\frac{r_i}{r^3}(\vec{r}\cdot\vec{p})} - \cancel{\hbar\frac{r_i}{r^3}} + \cancel{i\frac{r_i}{r^3}\vec{r}\cdot\vec{p}} - i\frac{1}{r}p_i + \cancel{\hbar\frac{r_i}{r^3}} \right).$$

$$= 0$$

!!OUF!!

R_i^2

$$[l_i, R_j] = \underbrace{\left[l_i, \frac{1}{2m}\varepsilon_{jkl}(p_k l_l - l_k p_l)\right]}_{A} + \underbrace{\left[l_i, \frac{r_j}{r}\right]}_{i\hbar\varepsilon_{ijk}\frac{r_k}{r}}$$

$$A = \frac{1}{2m}\varepsilon_{jkl}\left(l_i p_k l_l - l_i l_k p_l - p_k l_l l_i + l_k p_l l_i \right).$$

$$= \frac{1}{2m}\varepsilon_{jkl}\left(p_k l_i l_l + i\hbar\varepsilon_{ikm} p_m l_l - l_k l_i p_l - i\hbar\varepsilon_{ikm} l_m p_l - p_k l_l l_i + l_k p_l l_i \right).$$

$$= \frac{1}{2m}\varepsilon_{jkl}\left(i\hbar\varepsilon_{ilm} p_k l_m + i\hbar\varepsilon_{ikm} p_m l_l - i\hbar\varepsilon_{ilm} l_k p_m - i\hbar\varepsilon_{ikm} l \right.$$

$$= \frac{i\hbar}{2m}\left(\underbrace{(\varepsilon_{jlk}\varepsilon_{ikm} + \varepsilon_{jkm}\varepsilon_{ikl})}_{B} p_l l_m - (\varepsilon_{jlk}\varepsilon_{ikm} + \varepsilon_{jkm}\varepsilon_{ilm}) l_l p \right.$$

$$B = \delta_{jm}\delta_{il} - \cancel{\delta_{ji}\delta_{lm}} + \cancel{\delta_{ij}\delta_{lm}} - \delta_{jl}\delta_{im}.$$

$$= \varepsilon_{ijk}\,\varepsilon_{lmk}$$

$\cancel{[R_i, R_j] =}$

$$A = \frac{i\hbar}{2m} \varepsilon_{ijk} \varepsilon_{klm} (p_l \ell_m - \ell_l p_m)$$

$$= i\hbar \varepsilon_{ijk} \left(\frac{1}{2m} \varepsilon_{klm} (p_l \ell_m - \ell_l p_m) \right)$$

$\hookrightarrow$ $[\ell_i, R_j] = i\hbar \varepsilon_{ijk} R_k$.

7) $[k_i\, k_j] = \frac{m}{-2E} [R_i R_j] = i\hbar \varepsilon_{ijk} \ell_k$.

$$[\ell_i, k_j] = \sqrt{\frac{m}{-2E}} [\ell_i, R_j] = i\hbar \varepsilon_{ijk} \sqrt{\frac{m}{-2E}} \; R_k$$

$$= i\hbar \varepsilon_{ijk} k_k.$$

$$[\ell_i, \ell_j] = i\hbar \varepsilon_{ijk} \ell_k$$

8) $M \in SO(3)$: $\det M = +1$

$${}^tM\, M = \mathbb{1}$$

$$M = e^{i J_i \delta\theta_i} = \mathbb{1} + i \delta\theta_i J_i$$

$${}^tM M = (\mathbb{1} + i \delta\theta_i\, {}^tJ_i)(\mathbb{1} + i \delta\theta_j J_j).$$

$$= \mathbb{1} + i \delta\theta_i ({}^tJ_i + J_i) = \mathbb{1}.$$

$\hookrightarrow$ J_i antisymétrique.

$\hookrightarrow \frac{n(n-1)}{2}$ générateurs

$SO(2) \Rightarrow 1$

$SO(3) \Rightarrow 3$

$SO(4) \Rightarrow 6$

$$[J_{mn}, J_{pq}] = (J_{mn})_{ik}(J_{pq})_{kj} - \begin{pmatrix} m \leftrightarrow p \\ n \leftrightarrow q \end{pmatrix}$$

$$= -(\delta_{mi}\delta_{nk} - \delta_{mk}\delta_{ni})(\delta_{pk}\delta_{qj} - \delta_{pj}\delta_{qk}) + (\leftrightarrow)$$

$$= -(\delta_{mi}\delta_{np}\delta_{qj} - \delta_{mi}\delta_{nq}\delta_{pj} - \delta_{mp}\delta_{ni}\delta_{qj} + \delta_{mq}\delta_{ni}\delta_{pj})$$

$$+ (\delta_{pi}\delta_{qm}\delta_{nj} - \delta_{pi}\delta_{qn}\delta_{mj} - \delta_{pn}\delta_{qi}\delta_{nj} + \delta_{pn}\delta_{qi}\delta_{mj})$$

$$= \delta_{np}(\delta_{qi}\delta_{mj} - \delta_{mi}\delta_{qj}) + \delta_{mq}(\delta_{pi}\delta_{nj} - \delta_{ni}\delta_{pj})$$

$$- \delta_{mp}(-\delta_{ni}\delta_{qj} + \delta_{qi}\delta_{nj}) - \delta_{nq}(\delta_{pi}\delta_{mj} - \delta_{mi}\delta_{pj})$$

$$= i\{\delta_{np} J_{qm} + \delta_{mq} J_{pn} - \delta_{mp} J_{qn} - \delta_{nq} J_{pm}\}$$

$$= i(\delta_{mp} J_{nq} + \delta_{nq} J_{mp} - \delta_{mq} J_{np} - \delta_{np} J_{mq}).$$

§) $[a_i, a_j] = [J_{0i}, J_{0j}] = i(J_{ij}) = i\varepsilon_{ijk} b_k.$

$$[b_i, a_j] = \frac{1}{2}\varepsilon_{ikl}[J_{kl}, J_{0j}]$$

$$= \frac{i}{2}\varepsilon_{ikl}(-\delta_{kj} J_{l0} + \delta_{lj} J_{k0})$$

$$= \frac{i}{2}(+\varepsilon_{ijl} a_l - \varepsilon_{ikj} a_k)$$

$$= i\varepsilon_{ijk} a_k.$$

$$[b_i; b_j] = \frac{1}{4} \varepsilon_{ilm} \varepsilon_{jpq} [J_{lm}, J_{pq}]$$
$$= \frac{i}{4} \varepsilon_{ilm} \varepsilon_{jpq} \{ \delta_{lp} J_{mq} + \delta_{mq} J_{lp} - \delta_{lq} J_{mp} - \delta_{mp} J_{lq}$$
$$= \frac{i}{4} \{ - J_{ji} - J_{ji} - J_{ji} - J_{ji} = i J_{ij}$$
$$= i \varepsilon_{ijk} b_k.$$

$\vec{a}$ joue le même rôle que $\vec{k}/\hbar$
$\vec{b}$ " " $\vec{e}/\hbar$.

L'atome d'hydrogène présente une symétrie $SO(4)$! beaucoup plus grand que $SO(3)$...

10] $$[M_i; M_j] = \frac{1}{4} \left([l_i, l_j] + [l_i, k_j] + [k_i, l_j] + [k_i, k_j] \right)$$
$$= \frac{i\hbar}{4} \varepsilon_{ijk} \left(l_k + k_k + k_k + l_k \right)$$
$$= i\hbar \varepsilon_{ijk} M_k.$$

$$[M_i; N_j] = \frac{1}{4} \left([l_i, l_j] - [l_i, k_j] + [k_i, l_j] - [k_i, k_j] \right)$$
$$= \frac{i\hbar}{4} \varepsilon_{ijk} \left(l_k - k_k + k_k - l_k \right) = 0$$

$$[N_i; N_j] = \frac{i\hbar}{4} \varepsilon_{ijk} \left(l_k - k_k - k_k + l_k \right)$$
$$= i\hbar \varepsilon_{ijk} N_k.$$

↳ les M_i et les N_j commutent entre eux, et forment chacun l'algèbre de $SU(2)$

↳ $SO(4) \sim SU(2) \times SU(2)$
$\sim SO(3) \times SO(3)$.

$$M^2 = \frac{1}{4}\left(\ell^2 + h^2 + \underset{0}{\underset{\shortparallel}{\vec{\ell}\vec{h}}} + \underset{0}{\underset{\shortparallel}{\vec{h}\vec{\ell}}}\right) \quad \text{cf. } 4)$$

$$N^2 = \frac{1}{4}\left(\ell^2 + h^2 - \vec{\ell}\vec{h} - \vec{h}\cdot\vec{\ell}\right)$$

$$= \frac{1}{4}\left(\ell^2 + h^2\right) = M^2.$$

~~un~~ vecteur propre de M^2: $M^2|nm\rangle = \tilde{N}^2|nm\rangle$.

$$m|n\rangle = N^2|n\rangle$$

$\Rightarrow$ vecteur propre de N^2.

12)

$$h^2 = -\frac{m}{2E} R^2.$$

$\cancel{h^2 + \ell^2}$

$$h^2 + \ell^2 = 4M^2.$$

$$-\frac{m}{2E}\left(e^4 + \frac{2E(\ell^2 + \hbar^2)}{m}\right) + \ell^2 = 4M^2.$$

$$-\frac{me^4}{2E} - \hbar^2 = 4M^2$$

$$-\frac{me^4}{2E} = \hbar^2\left(4h(h+1) + 1\right)$$

$$= \hbar^2(2h+1)^2$$

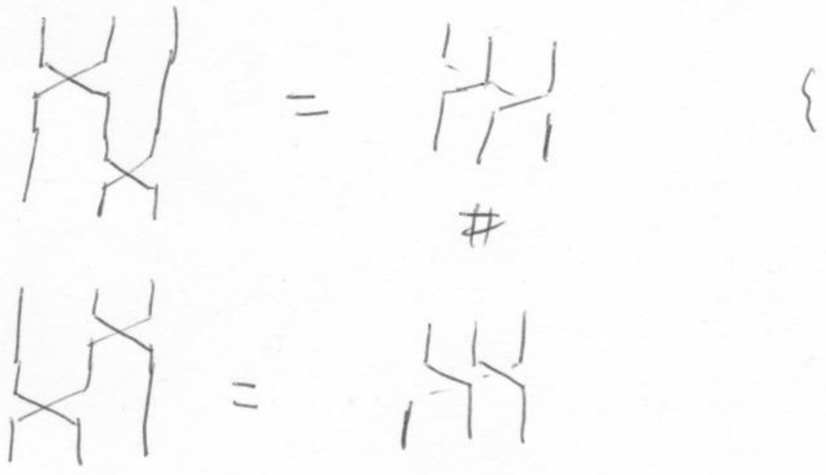
$$E = -\frac{me^4}{2\hbar^2(2h+1)^2} = -\frac{R_H}{(2h+1)^2} \quad \text{où } R_H = \frac{me^4}{2\hbar^2}.$$

1) * Loi interne.

* identité : $\{123\} \times g = g \times \{123\} = g$.
* associativité OK.
* inverse : OK.
*

- il y a 6 éléments $\{123\}$ $\{213\}$ $\{312\}$ $\{132\}$ $\{231\}$ $\{321\}$. $\Rightarrow$

- groupe non commutatif :



2) $\{123\} : A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $\{213\} : A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\{312\} : A_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$\{132\} : A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\{231\} : A_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ $\{321\} : A_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$A_i \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

↳ ~~les~~ vecteurs $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ ne se transforme pas : représentation triviale fermée

3) on introduit $M_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$M_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$M_3 = \sqrt{\frac{2}{3}} \begin{pmatrix} 1/2 \\ +\frac{1}{2} \\ -1 \end{pmatrix}$

o vérifie facilement que $\vec{M}_i \cdot \vec{M}_j = \delta_{ij}$

on construit $S = \left(\vec{M}_1, \vec{M}_2, \vec{M}_3\right)$

$$A'_i = {}^tS \; A_i \; S.$$

$$\left(A'_i\right)_{11} = 1$$

$\left(A'_i\right)_{\cancel{1a}}$ et $\left(A'_i\right)_{a\cancel{1}}$ $\quad a = 2,3$ sont obtenus comme un produit scalaire

entre $\underbrace{\vec{M}_1 \text{ avec les coordonnées permutées}}_{=\vec{M}_1}$ et $\vec{M}_2, \vec{M}_3$

$\hookrightarrow \left(A'_i\right)_{1a} = \left(A'_i\right)_{a,1} = 0. \Rightarrow$ matrice bloc diagonale

4) Les 6 matrices correspondent aux rotations le long de $(1,1,1)$

d'angle $\pm \frac{2\pi}{3}$

et aux images miroir / aux plans (xOy), (yOz) et (zOx).

ces opérations laissent (111) inchangé.

~~Les compos~~ Les parties orthogonale à cette direction se transforment

entre elles.

1) $T'_{ii} = R_{ik} R_{il} T_{kl}$

$R \in SO(3) \Rightarrow {}^tR R = \mathbb{1}$

$T'_{ii} = \delta_{kl} T_{kl} = T_{kk}$.

↳ la trace est invariante.

↳ la transformée de la trace ne ~~dépend~~ fait intervenir que de la trace ⇒

représentation (triviale) du groupe de rotation.

2) $A'_{ij} = \frac{1}{2} \left(R_{ik} R_{jl} T_{kl} - R_{jk} R_{il} T_{kl} \right)$

$= \frac{1}{2} R_{ik} R_{jl} \left(T_{kl} - T_{lk} \right)$

$= R_{ik} R_{jl} A_{kl}$.

La transformée de A ne fait intervenir que A ⇒ représentation irréductible

$V'_i = \frac{1}{2} \varepsilon_{ijk} R_{jl} R_{km} A_{lm}$.

$\varepsilon_{ijk} R_{jl} R_{km} = \varepsilon_{ijk} \left(\delta_{jl} + i\theta_\alpha J^\alpha_{jl} \right) \left(\delta_{km} + i\theta_\beta J^\beta_{km} \right)$

$= \varepsilon_{ilm} + i\theta_\alpha \left(\varepsilon_{ilk} (-i) \varepsilon_{k\alpha km} + \varepsilon_{ijm} (-i) \varepsilon_{\alpha jl} \right)$

$+ o(\theta^2)$

$= \varepsilon_{ilm} + i\theta_\alpha (-i) \left(\delta_{im} \delta_{l\alpha} - \cancel{\delta_{i\alpha} \delta_{lm}} + \cancel{\delta_{i\alpha} \delta_{ml}} - \delta_{il} \delta_{m\alpha} \right)$

$= \varepsilon_{ilm} + i\theta_\alpha (-i)$ $\varepsilon_{i\alpha k}$ ε_{mlk}.

$V'_i = V_i + i\,\theta_\alpha(-i)\ \varepsilon_{ik\alpha}\,V_k$.

↳ se transforme comme un vecteur ⇒

A : représentation vectorielle du groupe des rotations.

3] $S'_{ij} = \frac{1}{2}\left(T'_{ij} + T'_{ji}\right) = \frac{1}{2}\left(R_{ik}R_{je}\,T_{ke} + R_{jk}R_{ie}\,T_{ke}\right)$.

$= R_{ik}R_{je}\quad S_{ke}$.

4] générateur des rotations autour de x :

$S'_{ij} = S_{ij} + i\theta(-i)\left(\delta_{ik}\,\varepsilon_{1je} + \delta_{je}\,\varepsilon_{1ik}\right) S_{ke}$.

$S'_{11} = \qquad (\ 0\ )$.

$S'_{12} = \qquad (\ S_{13})$

$S'_{13} = \qquad (-S_{12})$

$S'_{22} = \qquad (\ 2S_{23})$

$S'_{23} = \qquad (-S_{22}+S_{33})$

$S'_{33} = \qquad (-2S_{23})$

$$J_x = -i \begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

rotations autour de y:

S'_{22}	$\to$	0
S'_{23}		S_{12}
S'_{12}		$-S_{23}$
S'_{33}		$2\,S_{13}$
S'_{13}		$S_{11}-S_{33}$
S'_{11}		$-2\,S_{13}$

$$J_y = -i \begin{pmatrix} 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

rotations autour de z:

S'_{33}	0
S'_{13}	S_{23}
S'_{23}	$-S_{13}$
S'_{11}	$2S_{12}$
S'_{12}	$S_{22}-S_{11}$
S'_{22}	$-2S_{12}$.

$$J_z = -i \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

On retrouve les relations de commutation...

Ⓐ

ANNEXE: Calcul des relations supplémentaires...

$[R_i, R_j]$? on calculera $[R_x, R_y]$

$$R_x = \frac{1}{m}\left(P_y l_z - P_z l_y - i\hbar P_x\right) - e^2 \frac{x}{r}$$

$$R_y = \frac{1}{m}\left(P_z l_x - P_x l_z - i\hbar P_y\right) - e^2 \frac{y}{r}$$

$$[R_x, R_y] = \frac{1}{m^2}\left[P_y l_z - P_z l_y - i\hbar P_x , P_z l_x - P_x l_z - i\hbar P_y\right]$$
$$- \frac{e^2}{m}\left\{\left[P_y l_z - P_z l_y - i\hbar P_x , \frac{y}{r}\right] - \left[P_z l_x - P_x l_z - i\hbar P_y , \frac{x}{r}\right]\right\}$$

$$[P_y l_z , P_z l_x] = P_y l_z P_z l_x - P_z l_x P_y l_z$$
$$= P_z P_y l_x l_z + i\hbar P_z P_y l_y - P_z l_x P_y l_z$$
$$= -i\hbar P_z^2 l_z + i\hbar P_z P_y l_y$$

$$[P_y l_z , P_x l_z] = P_y l_z P_x l_z - P_x l_z P_y l_z$$
$$= P_y P_x l_z l_z + i\hbar P_y^2 l_z - P_x l_z P_y l_z$$
$$= i\hbar P_x^2 l_z + i\hbar P_y^2 l_z$$

~~$[P_z l_y , P_x l_z] =$~~

$$[P_z l_y , P_z l_x] = P_z l_y P_z l_x - P_z l_x P_z l_y$$
$$= i\hbar P_z l_y P_y + P_z l_y l_x P_z - P_z l_x P_z l_y$$
$$= i\hbar P_z l_y P_y + P_z l_x l_y P_z - i\hbar P_z l_z P_z - P_z l_x P_z l_y$$
$$= i\hbar\left(P_z P_y l_y + P_x P_z l_x - P_z^2 l_z\right)$$

$$[P_z l_y , P_x l_z] = P_z l_y P_x l_z - P_x l_z P_z l_y$$
$$= P_z P_x l_y l_z - i\hbar P_z^2 l_z - P_x l_z P_z l_y$$
$$= P_z P_x \cancel{l_z} l_y + i\hbar P_z P_x l_x - i\hbar P_z^2 l_z - P_x \cancel{l_z P_z l_y}$$
$$= i\hbar\left(P_z P_x l_x - P_z^2 l_z\right)$$

Le premier commutateur ;

B

$$
\begin{aligned}
i\hbar \Big\{ & -\cancel{P_z^2 \ell_z}^{(7)} + \cancel{P_y P_z \ell_y}^{(2)} - \cancel{P_x^2 \ell_z}^{(3)} - \cancel{P_y^2 \ell_z}^{(4)} - \cancel{P_y P_z \ell_y}^{(2)} - \cancel{P_x P_z \ell_x}^{(6)} + \cancel{P_z^2 \ell_z}^{(7)} \\
& + \cancel{P_x P_z \ell_x}^{(6)} - P_z^2 \ell_z - \cancel{P_x P_z \ell_x}^{(5)} + \cancel{P_z \ell_x P_x}^{(5)} + \cancel{P_x^2 \ell_z}^{(3)} - P_x \ell_z P_x \\
& - P_y \ell_z P_y + \cancel{P_y^2 \ell_z}^{(4)} + \cancel{P_z P_y \ell_y}_{(1)} - \cancel{P_y P_z \ell_y}_{(1)} \Big\}.
\end{aligned}
$$

$$= i\hbar \left\{ -P_z^2 \ell_z - P_x^2 \ell_z - i\hbar \cancel{P_x P_y} + i\hbar \cancel{P_y P_x} - P_y^2 \ell_z \right\}$$

$$= -i\hbar P^2 \ell_z.$$

$$
\begin{aligned}
[P_x, r^2] &= P_x x x - x x P_x \\
&= -i\hbar x + x P_x x - x x P_x \\
&= -2 i\hbar x
\end{aligned}
$$

$$[P_x, (r^2)^\alpha] = -2 i \alpha \hbar x (r^2)^{\alpha-1}$$

$$[P_x, f(r^2)] = -2 i \hbar x f'(r^2).$$

$$[P_x, \tfrac{1}{r}] = i\hbar \frac{x}{r^3}.$$

$$
\begin{aligned}
[P_y \ell_z, \tfrac{y}{r}] &= P_y \ell_z y \frac{1}{r} - y \frac{1}{r} P_y \ell_z \\
&= P_y y \frac{1}{r} \ell_z - i\hbar \cancel{P_y} x \frac{1}{r} - y \frac{1}{r} P_y \ell_z \\
&= y P_y \frac{1}{r} \ell_z - i\hbar P_y \frac{x}{r} - y \frac{1}{r} P_y \ell_z - i\hbar \frac{1}{r} \ell_z \\
&= \frac{i y^2 \hbar}{r^3} \ell_z - i\hbar \frac{1}{r} \ell_z - i\hbar P_y \frac{x}{r}.
\end{aligned}
$$

$$
\begin{aligned}
[P_z \ell_y, \tfrac{y}{r}] &= P_z \ell_y y \frac{1}{r} - y \frac{1}{r} P_z \ell_y. \\
&= y \frac{i\hbar z}{r^3} \ell_y
\end{aligned}
$$

$$
\begin{aligned}
[P_z \ell_x, \tfrac{x}{r}] &= P_z \ell_x \frac{x}{r} - \frac{x}{r} P_z \ell_x \\
&= i\hbar \frac{x z}{r^3} \ell_x
\end{aligned}
$$

$$
\begin{aligned}
[P_x \ell_z, \tfrac{x}{r}] &= P_x \ell_z x \frac{1}{r} - x \frac{1}{r} P_x \ell_z. \\
&= P_x x \ell_z \frac{1}{r} + i\hbar P_x y \frac{1}{r} - x \frac{1}{r} P_x \ell_z. \\
&= x P_x \ell_z \frac{1}{r} - i\hbar \ell_z \frac{1}{r} + i\hbar P_x y \frac{1}{r} - x \frac{1}{r} P_x \ell_z. \\
&= i\hbar \frac{x^2}{r^3} \ell_z - i\hbar \frac{1}{r} \ell_z + i\hbar P_x y \frac{1}{r}
\end{aligned}
$$

2ème + 3ème commutateurs,

$$\frac{i\hbar}{r}\left\{ \frac{y^2}{r^2} \ell_z - \ell_z - r\cancel{p_y}\frac{x}{r} - \frac{yz}{r^2}\ell_y - \frac{xz}{r^2}\ell_x + \frac{x^2}{r^2}\ell_z - \ell_z + r\,\cancel{p_x y}\frac{1}{r} \right\}$$

$$+ i\hbar\left\{ - \cancel{p_x \frac{y}{r}} + \frac{y}{r}p_x + \cancel{p_y \frac{x}{r}} - \frac{x}{r}p_y \right\}.$$

$$yz\ell_y + xz\ell_x = yzzp_x - \cancel{yzxp_z} + \cancel{xzyp_z} - xz^2p_y$$
$$= -z^2\ell_z.$$

$$\hookrightarrow \frac{i\hbar}{r}\left\{ \ell_z - \ell_z - \ell_z - \ell_z \right\} = -\frac{2i\hbar}{r}\ell_z.$$

$$\hookrightarrow [R_x R_y] = \frac{-i\hbar p^2}{m^2}\ell_z + \frac{e^2}{m}2i\hbar\frac{1}{r}\ell_z$$
$$= -\frac{2i\hbar}{m}\left(\frac{p^2}{2m} - \frac{e^2}{r}\right)\ell_z$$
$$= -\frac{2i\hbar}{m}H\ell_z$$

$$\hookrightarrow [R_i, R_j] = -\frac{2i\hbar}{m}H\varepsilon_{ijk}\ell_k. \qquad \text{OUF!}$$

$$R_x = \frac{1}{m}\left(p_y l_z - p_z l_y\right) - \frac{i\hbar}{m} p_x - e^2 \frac{x}{r}.$$

$$R_x^2 = \frac{1}{m^2}\overbrace{\left(p_y l_z - p_z l_y\right)\left(p_y l_z - p_z l_y\right)}^{A} - \frac{\hbar^2}{m^2} p_x^2 + e^4 \frac{x^2}{r^2}$$
$$- \frac{i\hbar}{m^2}\overbrace{\left\{ p_x , p_y l_z - p_z l_y \right\}}^{B} - \frac{e^2}{m}\overbrace{\left\{ p_y l_z - p_z l_y , \frac{x}{r} \right\}}^{C}$$
$$+ \frac{i\hbar e^2}{m}\overbrace{\left\{ p_x , \frac{x}{r} \right\}}^{D}.$$

$$A = p_y l_z p_y l_z + p_z l_y p_z l_y - p_y l_z p_z l_y - p_z l_y p_y l_z$$
$$= p_y^2 l_z^2 + p_z^2 l_y^2 - i\hbar\, p_y p_x l_z + i\hbar\, p_z p_x l_y - \underbrace{\left\{ p_y(x p_y - y p_x), p_z(z p_x - x \right.}_{E}$$

$$E = \underbrace{p_y x p_y p_z z}_{a} p_x + p_z \underbrace{z p_x p_y x}_{a} p_y + \underbrace{p_y y p_x p_z x p_z}_{b} + p_z \underbrace{x p_z p_y y p_x}_{b}$$
$$- \underbrace{p_y x p_y p_z x}_{a} p_z - p_z \underbrace{x p_z p_y x}_{b} p_y - p_y \underbrace{y p_x p_z z}_{c} p_x - p_z \underbrace{z p_x p_y y p_x}_{c}$$
$$- \underbrace{p_y^2 z p_x z}_{a} p_x + \underbrace{p_y^2 z p_x z p_x}_{c} - p_z^2 \underbrace{y p_x y}_{b} p_x + \underbrace{p_z^2 y p_x y}_{c} p_x$$
$$= -p_y^2\left(x p_z x p_z + z p_x z p_x - x p_z z p_x - z p_x x p_z + i\hbar\, p_x x \right)$$
$$- p_z^2\left(x p_y x p_y + y p_x y p_x - x p_y y p_x - y p_x x p_y + i\hbar\, p_x x \right)$$
$$+ p_x^2\left(z p_y z p_y + y p_z y p_z - y p_z z p_y + i\hbar\, p_z z - z p_y y p_z + i\hbar\, p_y y \right)$$
$$= -p_y^2 l_y^2 - p_z^2 l_z^2 + p_x^2 l_x^2 + i\hbar\left(-p_y^2 p_x x - p_z^2 p_x x + p_x^2 p_z z + p_x^2 p_y y \right)$$

$$A = p_y^2 l_z^2 + p_z^2 l_y^2 + p_y^2 l_y^2 + p_z^2 l_z^2 - p_x^2 l_x^2 + i\hbar \left\{ \cancel{p_x p_y} z p_x - \cancel{p_x p_z x p_z} \right.$$
$$\left. - p_x \cancel{p_y x} p_y + p_x \cancel{p_y y} p_x + \cancel{p_y^2 p_x x} + \cancel{p_z^2 p_x x} - \cancel{p_x^2 p_z z} - \cancel{p_x^2 p_y y} \right\}$$

$$A = (P_y^2 + P_z^2)(\ell_y^2 + \ell_z^2) - P_x^2 \ell_x^2$$

$$\begin{aligned} B = \;& P_x P_y x P_y + P_y x P_y P_x - P_x P_y y P_x - P_y y P_x P_x \\ & - P_x P_z z P_x - P_z z P_x P_x + P_x P_z x P_z + P_z x P_z P_x \\ = \;& 2 P_y^2 P_x x + i\hbar P_y^2 - 2P_x^2 P_y y - 2P_x^2 P_z z + 2P_z^2 P_x x + i\hbar P_z^2 \\ = \;& 2\Big(\underbrace{(P_y^2 + P_z^2) P_x x - P_x^2 (P_y y + P_z z)}_{\Sigma R_i^2 \ell \Rightarrow 0.}\Big) + i\hbar (P_y^2 + P_z^2) \end{aligned}$$

$$[P_x, r^2] = P_x r^2 - x^2 P_x = -i\hbar x + x P_x x - x^2 P_x$$
$$= -2i\hbar x$$

$$[P_x, f(r^2)] = -2i\hbar x f'(r^2)$$
$$[P_x, \tfrac{1}{r}] = i\hbar \frac{x}{r^3}$$

$$\begin{aligned} C = \;& P_y \ell_z \frac{x}{r} + \frac{x}{r} P_y \ell_z - P_z \ell_y \frac{x}{r} - \frac{x}{r} P_z \ell_y \\ = \;& x P_y \frac{1}{r} \ell_z + i\hbar P_y \frac{y}{r} + \frac{x}{r} P_y \ell_z - x P_z \frac{1}{r} \ell_y + i\hbar P_z \frac{z}{r} - \frac{x}{r} P_z \ell_y \\ = \;& \frac{2}{r}(x P_y \ell_z - x P_z \ell_y) + i\hbar \left(\frac{xy}{r^3} \ell_z + P_y \frac{y}{r} - \frac{xz}{r^3} \ell_y + P_z \frac{z}{r} \right) \end{aligned}$$

$$\frac{-e^2}{m} C + \frac{i\hbar}{m} e^2 D$$

$$= \frac{e^2}{m} \left\{ -\frac{2}{r}(x P_y \ell_z - x P_z \ell_y) + i\hbar \left(-\frac{xy}{r^3} \ell_z + \frac{xz}{r^3} \ell_y - P_y \frac{y}{r} - P_z \frac{z}{r} + P_x \frac{x}{r} + \frac{x}{r} P_x \right) \right.$$

$$\frac{x}{r} P_x = \frac{1}{r} P_x x + i\hbar \frac{1}{r}$$
$$= P_x \frac{x}{r} - i\hbar \frac{x^2}{r^3} + i\hbar \frac{1}{r}.$$

F

$$-\frac{e^2}{m}C + \frac{i\hbar}{m}e^2 D = \frac{e^2}{m}\Big\{-\frac{2}{r}\left(x P_y \ell_z - x P_z \ell_y\right) + i\hbar\Big(\overbrace{-\frac{xy}{r^3}\ell_z + \frac{xz}{r^3}\ell_y}^{\Sigma R_i^2 = 0}$$

$$\underbrace{-P_y\frac{y}{r} - P_z\frac{z}{r} + 2P_x\frac{x}{r}}_{\Sigma_i R_i^2 = 0}\Big) - \hbar^2\left(\frac{1}{r} - \frac{x^2}{r^3}\right)\Big\}$$

$$\sum_i R_i^2 = \frac{1}{m^2}P^2\ell^2 - \frac{\hbar^2}{m^2}P^2 + e^4 + \frac{\hbar^2}{m^2}2P^2 + \frac{e^2}{m}\left\{-\frac{2}{r}\ell^2 - \hbar^2\frac{2}{r}\right\}$$

$$= \frac{2}{m}\left\{\frac{1}{2m}P^2(\ell^2+\hbar^2) - \frac{e^2}{r}(\ell^2+\hbar^2)\right\} - e^4$$

$$R^2 = \frac{2H}{m}(\ell^2+\hbar^2) + e^4$$

OUF!