Feuille 1

[I]

1) $\underline{\delta_{ii} = 3}$

$\underline{\delta_{ik}\delta_{kj} = \delta_{ij}}$

$\underline{\varepsilon_{iik} = 0}$

2) j,k sont différents, et différents de i
l,m — — — } ⇒ contribution non nulle.

↳ Soit $j = l \neq k = m$
soit $j = m \neq k = l$

par inspection :

$$\underline{\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}}$$

$$\varepsilon_{ijk}\varepsilon_{ijl} = \delta_{jj}\delta_{kl} - \delta_{jl}\delta_{kj} = 3\delta_{kl} - \delta_{kl} = \underline{2\delta_{kl}}$$

$$\varepsilon_{ijk}\varepsilon_{ijk} = 2\delta_{kk} = \underline{6}$$

3) $$\vec{x}\wedge\vec{y} = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} \times \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = \begin{vmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{vmatrix}$$

$$\underline{\vec{x}\wedge\vec{y}\,\big|_i = \varepsilon_{ijk}\, x_j\, y_k}$$

$$\vec{x}\cdot(\vec{y}\wedge\vec{z}) = x_i\varepsilon_{ijk}\, y_j\, z_k = \underline{\varepsilon_{ijk}\, x_i\, y_j\, z_k}$$

$$\varepsilon_{ijk} = -\varepsilon_{jik} = \varepsilon_{jk\dots}$$

$$= \varepsilon_{jki}\, x_j\, z_k\, x_i$$

$$= \varepsilon_{ijk}\, y_i\, z_j\, x_k$$

$$= \vec{y}\cdot(\vec{z}\wedge\vec{x})$$

change le nom des variables (muettes)

4) $A_{ii} = \mathrm{Tr}\, A$

$A_{ij}B_{jk}$ est la composante ik de $A.B$.

$A_{ij}B_{ij} = \mathrm{Tr}({}^tA\,B)$

il y a 6 choix de (ijk) donnant une contribution non nulle. La première :
$(ijk)=(123) \Rightarrow$

$$\varepsilon_{123} A_{1l} A_{2m} A_{3n} \varepsilon_{lmn} = A_{11} A_{22} A_{33} - A_{11} A_{23} A_{32} + A_{12}(A_{23} A_{31} - A_{21} A_{33}) + A_{13}(A_{21} A_{32} - A_{22} A_{31})$$
$$= \det A$$

Pour les 5 autres possibilités, après permutation et changement de nom, on retrouve le même résultat :

$$\varepsilon_{213} A_{2l} A_{1m} A_{3n} \varepsilon_{lmn} = (-\varepsilon_{123}) A_{1m} A_{2l} A_{3n} (-\varepsilon_{mln}) = A_{1l} A_{2m} A_{3n} \varepsilon_{lmn}$$

$$\Rightarrow \frac{1}{6} \varepsilon_{ijk} \varepsilon_{lmn} A_{il} A_{jm} A_{kn} = \det A.$$
$$= \varepsilon_{lmn} A_{1l} A_{2m} A_{3n}$$

5) $A_{ij} S_{ij} = - A_{ji} S_{ij} = - A_{ji} S_{ji} \underset{\text{changement de nom}}{=} - A_{ij} S_{ij} = 0$

$\varepsilon_{ijk} S_{jk} = - \varepsilon_{ijk} S_{jk} = 0$

> Généralement : sommation sur indices symétriques et antisymétriques $\Rightarrow 0$

6) $\vec{\nabla} V : \partial_i V$

$\vec{\nabla} \cdot \vec{A} : \partial_i A_i$

$\vec{\nabla} \times \vec{A} : \varepsilon_{ijk} \partial_j A_k$

$\Delta V : \partial_i \partial_i V$

$\Delta \vec{A} : \partial_i \partial_i A_j$

7) $\vec{\nabla} \times (\vec{\nabla} V) = \underbrace{\varepsilon_{ijk}}_{\text{Antisym}} \underbrace{\partial_j \partial_k}_{\text{Sym}} V = 0$

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \underbrace{\varepsilon_{ijk}}_{\text{Antisym.}} \underbrace{\partial_i \partial_j}_{\text{Sym}} A_k = 0$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \varepsilon_{ijk}\,\partial_j\,\varepsilon_{klm}\,\partial_l A_m$$
$$= (\varepsilon_{kij}\varepsilon_{klm})\,\partial_j\partial_l A_m$$
$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(\partial_j\partial_l A_m)$$
$$= \partial_i(\partial_j A_j) - \partial_j\partial_j A_i$$
$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \Delta\vec{A}$$

$\boxed{2}$ 1) $x_\mu = g_{\mu\nu}\,x^\nu = g_{\mu\nu}\,g^{\nu\rho}\,x_\rho \Rightarrow g_{\mu\nu}\,g^{\nu\rho} = \text{identité}$

$g^{\mu\nu}$: composantes de la matrice inverse :

$g^{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$, $g_{\mu\nu} = 0$ si $\mu \neq \nu$

2) $d\tau^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu$

$$= g_{\mu\nu}\,\Lambda^\mu{}_\rho\,dx^\rho\,\Lambda^\nu{}_\sigma\,dx^\sigma$$

$\hookrightarrow g_{\mu\nu}\,\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = g_{\rho\sigma}$ $(*)$

3) Soit $G = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$ $\Rightarrow {}^t\Lambda\, G\, \Lambda = G$

$L = (\Lambda^\mu{}_\rho)$

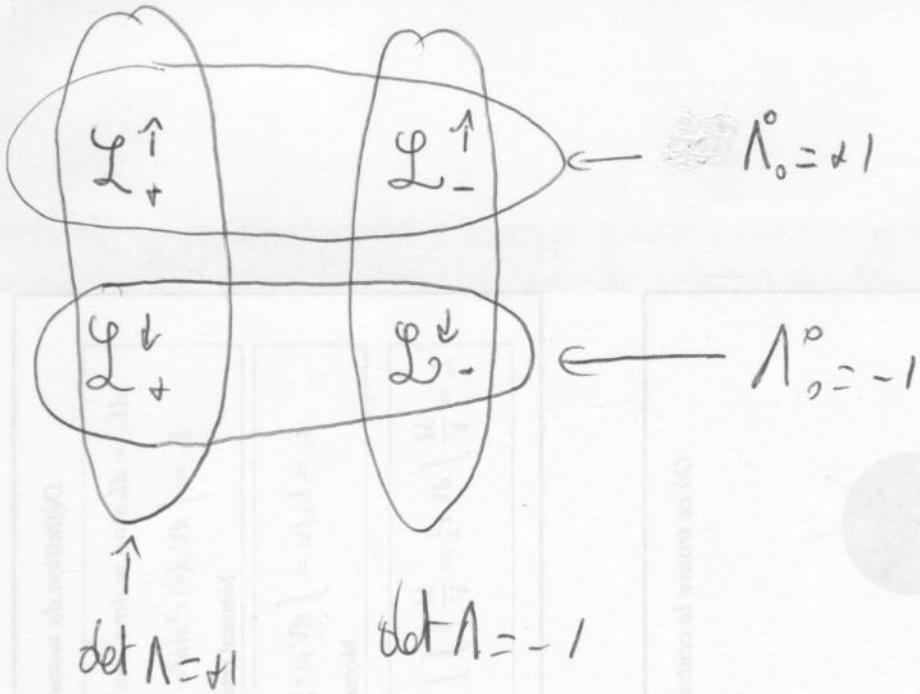
$\det G = \det({}^t\Lambda\, G\,\Lambda) = \det G \times \det{}^t\Lambda \times \det\Lambda = \det G\,(\det\Lambda)^2$

$\hookrightarrow (\det\Lambda)^2 = 1 \Leftrightarrow \det\Lambda = \pm 1$

relation $(*)$ pour $\rho = \sigma = 0$

$$g_{\mu\nu}\,\Lambda^\mu{}_0\Lambda^\nu{}_0 = 1 = {\Lambda^0{}_0}^2 - \left[(\Lambda^1{}_0)^2 + (\Lambda^2{}_0)^2 + (\Lambda^3{}_0)^2\right]$$

$\hookrightarrow {\Lambda^0{}_0}^2 \geq 1 \Rightarrow \Lambda^0{}_0 \geq 1$ ou $\Lambda^0{}_0 \leq -1$



$$\mathcal{L}_+^\uparrow \xrightarrow{PT} \mathcal{L}_+^\downarrow$$

$$\xrightarrow{P} \mathcal{L}_-^\uparrow$$

$$\xrightarrow{T} \mathcal{L}_-^\downarrow$$

4] Λ^{-1} est la transformation inverse:

$$\Lambda^{-1\mu}{}_\nu \Lambda^\nu{}_\rho = \delta^\mu{}_\rho$$

$$(*) : g^{\rho\alpha} g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \Lambda^{-1\sigma}{}_\tau = g_{\rho\sigma} \Lambda^{-1\sigma}{}_\tau \, g^{\rho\alpha}$$

$$\hookrightarrow \quad \Lambda^{-1\alpha}{}_\tau = g^{\rho\alpha} g_{\mu\tau} \Lambda^\mu{}_\rho = \Lambda_\tau{}^\alpha \qquad (**)$$

5] Pour qu'un sous ensemble H de G avec la loi de composition soit un sous groupe de G, il faut que:

1* l'identité soit dans le sous ensemble. ($1 \in H$).

2* la loi de composition soit interne ($\forall a, b \in H, ab \in H$)

3) l'inverse de tout élément de H soit dans H ($\forall a \in H$, $a^{-1} \in H$)

Il est satisfait pour les 3 sous-ensembles.

Pour $\mathcal{L}_+$: Soient $\Lambda, \Lambda' \in \mathcal{L}_+$

$$\det(\Lambda\Lambda') = \det\Lambda \det\Lambda' = 1 \Rightarrow \Lambda\Lambda' \in \mathcal{L}_+ \Rightarrow 2)$$

$$\det(\mathbb{1}) = \det(\Lambda\Lambda^{-1}) = \det\Lambda \det\Lambda^{-1} = \det\Lambda^{-1} = 1 \Rightarrow \Lambda^{-1} \in \mathcal{L}_+ \Rightarrow 3)$$

$\mathcal{L}_+$ est un sous groupe

Pour $\mathcal{L}^\uparrow$: Soient $\Lambda, \Lambda' \in \mathcal{L}^\uparrow$

$$\Lambda^{-1\,0}{}_0 = \Lambda_0{}^0 = g_{0\mu} g^{0\nu} \Lambda^\mu{}_\nu = \Lambda^0{}_0 \geqslant 1 \Rightarrow \Lambda^{-1} \in \mathcal{L}^\uparrow \Rightarrow 3)$$

En utilisant (*), $1 = \Lambda^0{}_0{}^2 - \sum_i \Lambda^i{}_0{}^2 \hookrightarrow \sum_i (\Lambda^i{}_0)^2 < \Lambda^0{}_0{}^2$.

D'autre part, $\Lambda^{-1} \in \mathcal{L}^\uparrow \Rightarrow \sum_i (\Lambda^{-1\,i}{}_0)^2 < (\Lambda^{-1\,0}{}_0)^2$

$$\hookrightarrow \sum_i (\Lambda^0{}_i)^2 < \Lambda^0{}_0{}^2 \quad \text{(en utilisant (**))}$$

Maintenant : $(\Lambda\Lambda')^0{}_0 = \Lambda^0{}_\mu \Lambda'^\mu{}_0 = \Lambda^0{}_0 \Lambda'^0{}_0 - \sum_i \Lambda^0{}_i \Lambda'^i{}_0$

On peut voir le 2ème terme comme le produit scalaire entre le vecteur à 3 composantes de $(\Lambda^0{}_i)$ de norme $< \Lambda^0{}_0$ et le vecteur $(\Lambda'^i{}_0)$ de norme $< \Lambda'^0{}_0$ $\Rightarrow$

$$-\Lambda^0{}_0 \Lambda'^0{}_0 < -\sum_i \Lambda^0{}_i \Lambda'^i{}_0 < \Lambda^0{}_0 \Lambda'^0{}_0$$

$$\hookrightarrow 0 < \Lambda^0{}_0 \Lambda'^0{}_0 - \sum_i \Lambda^0{}_i \Lambda'^i{}_0 = (\Lambda\Lambda')^0{}_0 \Rightarrow \Lambda\Lambda' \in \mathcal{L}^\uparrow$$

$\mathcal{L}^{\uparrow}$ est un sous groupe.

6] $x'_\mu = g'_{\mu\nu} x'^\nu = g_{\mu\nu} x'^\nu$ (g est un tenseur invariant le même dans tous les référentiels inertiels)

$= g_{\mu\nu} \Lambda^\mu{}_\rho x^\rho$

$= \underbrace{g_{\mu\nu} \Lambda^\nu{}_\rho g^{\rho\sigma}} x_\sigma$

$= \Lambda_\nu{}^\sigma x_\sigma = \Lambda^{-1\sigma}{}_\nu x_\sigma$

Les composantes covariantes se transforment avec la transformation inverse !!

7] $x'_\mu y'^\mu = \Lambda_\mu{}^\alpha x_\alpha \Lambda^\mu{}_\beta y^\beta = \Lambda^{-1\alpha}{}_\mu \Lambda^\mu{}_\beta \; x_\alpha y^\beta = x_\alpha y^\alpha$.

$x_\alpha y^\alpha$ Ne se transforme pas $\Rightarrow$ scalaire.

$A'^{\mu\nu} x'_\nu = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta A^{\alpha\beta} \Lambda^{-1\gamma}{}_\nu x_\gamma$

$= \Lambda^\mu{}_\alpha A^{\alpha\gamma} x_\gamma$ se transforme comme les composantes contravariantes d'un 4-vecteur.

$A'_{\mu\nu} x'^\mu y'^\nu = \Lambda^{-1\alpha}{}_\mu \Lambda^{-1\beta}{}_\nu \Lambda^\mu{}_\gamma \Lambda^\nu{}_\delta A_{\alpha\beta} x^\gamma y^\delta$.

$= A_{\gamma\delta} x^\gamma y^\delta$ scalaire.

Avec la convention d'Einstein on s'assure que les objets construits utilisés ont de bonnes propriétés de transformation.

8] $\partial'_\mu = \dfrac{\partial}{\partial x'^\mu} = \dfrac{\partial x^\rho}{\partial x'^\mu} \dfrac{\partial}{\partial x^\rho} = \dfrac{\partial \Lambda^{-1\rho}{}_\sigma x'^\sigma}{\partial x'^\mu} \partial_\rho$.

$= \Lambda^{-1\rho}{}_\sigma \partial_\rho$ Se transforme avec la matrice inverse.

Justifie que l'on mette l'indice en bas dans ∂_μ.

9] relation (*)
$$g_{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \, g_{\alpha\beta}$$
$$= (\delta^\alpha_\mu + \omega^\alpha{}_\mu)(\delta^\beta_\nu + \omega^\beta{}_\nu)\, g_{\alpha\beta}$$
$$= g_{\mu\nu} + \omega_{\nu\mu} + \omega_{\mu\nu} + O(\omega^2)$$

↳ $\omega_{\mu\nu} + \omega_{\nu\mu} = 0 \Rightarrow \omega_{\mu\nu}$ antisymétrique. $\Rightarrow$ 6 paramètres.

$$\omega_{\mu\nu} : \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \theta & 0 \\ 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \theta > 0.$$

$$\omega^\mu{}_\nu \doteq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta & 0 \\ 0 & \theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu$$

↳ $\begin{cases} t' = t \\ x' = x - \theta y \\ y' = y + \theta x \\ z' = z \end{cases}$ rotation infinitésimale.

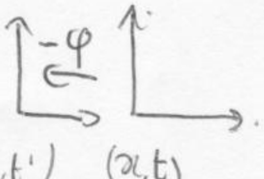
$y' = 0 :\ y = -\theta$

$$\omega_{\mu\nu} \doteq \begin{pmatrix} 0 & \varphi & 0 & 0 \\ -\varphi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\omega^\mu{}_\nu \doteq \begin{pmatrix} 0 & \varphi & 0 & 0 \\ \varphi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\begin{cases} t' = t + \varphi x \\ x' = x + \varphi t \\ y' = y \\ z' = z \end{cases}$ boost dans la direction x

$x' = 0 :\ x = -\varphi t$



10] $$\Lambda^\mu{}_\nu = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$(*) \Rightarrow \begin{matrix} a^2 - c^2 = 1 \\ b^2 - d^2 = -1 \\ ab - cd = 0 \end{matrix} \Rightarrow \begin{matrix} a = \operatorname{ch}\varphi & c = \operatorname{sh}\varphi \\ d = \operatorname{ch}\psi & b = \operatorname{sh}\psi \end{matrix}$

$\operatorname{ch}\varphi \operatorname{sh}\psi - \operatorname{ch}\psi \operatorname{sh}\varphi = 0$

$\operatorname{sh}(\psi - \varphi) = 0 \Rightarrow \psi = \varphi$

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \operatorname{ch}\varphi & \operatorname{sh}\varphi & 0 & 0 \\ \operatorname{sh}\varphi & \operatorname{ch}\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$t' = \operatorname{ch}\varphi\, t + \operatorname{sh}\varphi\, x$

$x' = \operatorname{sh}\varphi\, t + \operatorname{ch}\varphi\, x$

$y' = y$

$z' = z.$

$x' = 0 \Rightarrow x = -\operatorname{th}\varphi\, t.$

$v = -\operatorname{th}\varphi.$ $\qquad \operatorname{ch}\varphi = \frac{1}{\sqrt{1-v^2}}$ $\qquad \operatorname{sh}\varphi = -\frac{v}{\sqrt{1-v^2}}$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\boxed{3}$ 1) $L = T - V = \sum_{i=1}^{N} \frac{1}{2} m \dot{\varphi}_i^2 - \sum_{i=0}^{N} \frac{1}{2} k (\varphi_{i+1} - \varphi_i)^2$

$\varphi_0(t) = \varphi_{N+1}(t) = 0$

$S = \int dt\, L$

2) $S[\varphi + \delta\varphi] = S[\varphi] + \int dt \sum_{i=1}^{N} \underbrace{m \dot{\varphi}_i \, \delta\dot{\varphi}_i} - k \delta\varphi_i (2\varphi_i - \varphi_{i-1} - \varphi_{i+1}) + o(\delta$

Intégration par parties

$S[\varphi + \delta\varphi] - S[\varphi] = \sum_{i=1}^{N} \delta\varphi_i \int dt \left(-m \ddot{\varphi}_i + k (\varphi_{i+1} + \varphi_{i-1} - 2\varphi_i) \right)$

$\hookrightarrow m \ddot{\varphi}_i = k (\varphi_{i+1} + \varphi_{i-1} - 2\varphi_i) \quad \forall i$

3] Limite continue : $\varphi_{i+1} + \varphi_{i-1} - 2\varphi_i = \varphi(x+a) + \varphi(x-a) - 2\varphi(x)$

$= \cancel{\varphi(x)} + \cancel{a\varphi'(x)} + \frac{a^2}{2}\varphi''(x) + \frac{a^3}{6}\varphi'''(x)$
$\quad \cancel{\varphi(x)} - \cancel{a\varphi'(x)} + \frac{a^2}{2}\varphi''(x) - \frac{a^3}{6}\varphi'''(x) + o(a^4)$
$\quad - 2\cancel{\varphi(x)}$

$= a^2 \varphi''(x) + o(a^4)$

$\hookrightarrow m \ddot{\varphi}(x) = k a^2 \varphi''(x)$

$\mu \ddot{\varphi}(x) - Y \varphi''(x) = 0$

$\hookrightarrow \left(\partial_t^2 - c^2 \partial_x^2 \right) \varphi = 0$ avec $c^2 = \frac{Y}{\mu}$

4] $S[\varphi] = \int dt \sum_{i=1}^{N} a \frac{1}{2} \frac{m}{a} \dot{\varphi}_i^2 - \sum_{i=1}^{N} a \, \frac{1}{2}(ka) (\nabla\varphi)^2$

$\sum_{i=1}^{N} a\, X \sim \int dx\, X$

$$S[\varphi] = \int dx\,dt \; \frac{1}{2}\left(\mu(\partial_t\varphi)^2 - Y(\partial_x\varphi)^2\right)$$

$$S[\varphi+\delta\varphi] - S[\varphi] = \int dx\,dt \left(\mu\,\partial_t\varphi\,\partial_t\delta\varphi - Y\,\partial_x\varphi\,\partial_x\delta\varphi\right) + o(\delta\varphi^2)$$

Integration par parties:

$$\hookrightarrow \int dx\,dt\; \delta\varphi\left(-\mu\,\partial_t^2 + Y\,\partial_x^2\right)\varphi$$

$\hookrightarrow$ principe variationnel $\Rightarrow \left(-\mu\,\partial_t^2 + Y\,\partial_x^2\right)\varphi = 0 \iff \left(\partial_t^2 - c^2\,\partial_x^2\right)\varphi = 0$

5]
$$\frac{\delta \int_{xt} \phi(x,t)}{\delta \phi(x',t')} = 1$$

$$\frac{\delta \int_{xt} \phi(x,t)^n}{\delta \phi(x',t')} = \left\{\int_{xt}\left(\phi(x,t)^n + n\varepsilon\,\delta(x-x')\delta(t-t')\,\phi^{n-1}(x,t)\right) - \int_{xt}\phi(x,t)^n\right\} \Big/ \varepsilon$$

$$= n\,\phi^{n-1}(x',t')$$

$$\frac{\delta \int_{xt} \phi(x,t)^o\,\partial_t^m\,\partial_x^n\,\phi(x,t)}{\delta\phi(x',t')} = o\,\phi^{o-1}(x',t')\,\partial_t^m\,\partial_x^n\,\phi(x',t') + \int_{xt}\phi^o(x,t)\,\partial_t^m\,\partial_x^n\,\delta(x-x')\delta(t-t')$$

$$= o\,\phi^{o-1}(x',t')\,\partial_t^m\,\partial_x^n\,\phi(x',t') + (-1)^{m+n}\,\partial_{t'}^m\,\partial_{x'}^n\,\phi^o(x',t')$$

6) $S = \int_{xt} \frac{1}{2}\left(\mu(\partial_t\varphi)^2 - \gamma(\partial_x\varphi)^2\right)$

$$\frac{\delta S[\varphi]}{\delta\varphi(x',t')} = \int_{xt} \frac{1}{2}\left(2\mu(\partial_t\varphi)\,\partial_t\delta(t-t')\delta(x-x') - 2\gamma(\partial_x\varphi)\,\partial_x\delta(t-t')\delta(x-x')\right)$$

$$= \int_{xt}\left(-\mu\partial_t^2\varphi(x,t) + \gamma\partial_x^2\varphi(x,t)\right)\delta(t-t')\,\delta(x-x')$$

$$= \left(-\mu\partial_{t'}^2 + \gamma\partial_{x'}^2\right)\varphi(x',t')$$

7) $$L = \sum_{i=1}^{N}\frac{1}{2}m\dot\varphi_i^2 - \sum_{i=0}^{N}\frac{1}{2}k(\varphi_{i+1}-\varphi_i)^2 - \sum_{i=1}^{N}\frac{1}{2}k'\varphi_i^2$$

8) $$S[\varphi+\delta\varphi] - S[\varphi] = \int_t\left\{\sum_{i=1}^{N} m\dot\varphi_i\,\delta\dot\varphi_i - \sum_{i=1}^{N} k\,\delta\varphi_i(2\varphi_i - \varphi_{i-1} - \varphi_{i+1}) - \sum_{i=1}^{N} k'\varphi_i\delta\varphi_i\right\}$$

$$= \int_t\sum_{i=1}^{N}\delta\varphi_i\left(-m\ddot\varphi_i + k(\varphi_{i+1}+\varphi_{i-1}-2\varphi_i) - k'\varphi_i\right).$$

$\hookrightarrow\ m\ddot\varphi_i = k(\varphi_{i+1}+\varphi_{i-1}-2\varphi_i) - k'\varphi_i$

Limite continue :

$$m\ddot\varphi(x,t) = ka^2\,\partial_x^2\varphi - k'\varphi$$

$$\mu\partial_t^2\varphi = \gamma\,\partial_x^2\varphi - \gamma'\varphi \qquad \gamma' = \frac{k'}{a}$$

Klein-Gordon

9) $$S[\varphi] = \int_{xt}\frac{1}{2}\mu(\partial_t\varphi)^2 - \frac{1}{2}\gamma(\partial_x\varphi)^2 - \frac{1}{2}\gamma'\varphi^2$$

$$\frac{\delta S}{\delta\varphi(x',t')} = \int_{xt}\mu\partial_t\varphi\,\partial_t\delta(t-t')\,\delta(x-x') - \gamma\,\partial_x\varphi\,\partial_x\delta(x-x')\,\delta(t-t') - \gamma'\varphi\,\delta(x-x')\,\delta(t-t')$$

2

$$\left(-\mu \partial_{t'}^2 + Y \partial_{x'}^2 - Y'\right) \varphi(x',t') = 0.$$

$$\varphi(x,t) = \int \frac{dq}{2\pi} \frac{d\omega}{2\pi} \varphi(q,\omega) e^{i(\omega t - qx)}$$

$$\Rightarrow \mu\omega^2 - Y q^2 - Y' = 0$$

$$\omega^2 = \frac{Y}{\mu} q^2 + \frac{Y'}{\mu}$$ relation de dispersion d'une particule massive...

$\boxed{4}$ $\partial^\mu = (\partial_t, -\partial_{x_i})$

$$F^{0i} = \partial^0 A^i - \partial^i A^0$$

$$= \partial_t A^i + \partial_{x_i} V = \partial_t \vec{A} + \vec{\nabla} V$$

$$= -E_i$$

$$F^{i0} = E_i$$

$$F^{ij} = \partial^i A^j - \partial^j A^i = -\partial_{x_i} A_j + \partial_{x_j} A_i$$

$$F^{12} = \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} = -B_z$$

$$\Rightarrow F^{\mu\nu} = \begin{pmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & & \\ F^{20} & & & \\ F^{30} & & & \end{pmatrix} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma}$$

$$F'^{10} = E'_x = \Lambda^1{}_\rho \Lambda^0{}_\sigma F^{\rho\sigma}$$

$$= \Lambda^1{}_0 \Lambda^0{}_1 F^{01} + \Lambda^1{}_1 \Lambda^0{}_0 F^{10} = E_x\left(\Lambda^0{}_0\Lambda^1{}_1 - \Lambda^1{}_0\Lambda^0{}_1\right)$$

$$= E_x\left(\gamma^2 - \gamma^2 v^2\right) = E_x$$

$$F'^{20} = E'_y = \Lambda^2{}_2 \Lambda^0{}_0 F^{20} + \Lambda^2{}_2 \Lambda^0{}_1 F^{21}$$

$$= \gamma E_y - \gamma v B_z$$

$$F'^{30} = E'_z = \Lambda^3{}_3 \Lambda^0{}_0 F^{30} + \Lambda^3{}_3 \Lambda^0{}_1 F^{31}$$

$$= \gamma E_z + \gamma v B_y$$

$$F'^{32} = B'_x = \Lambda^3{}_3 \Lambda^2{}_2 F^{32} = B_x.$$

$$F'^{13} = B'_y = \Lambda^1{}_0 \Lambda^3{}_3 F^{03} + \Lambda^1{}_1 \Lambda^3{}_3 F^{13}$$

$$= +\gamma v E_z + \gamma B_y$$

$$F'^{21} = B'_z = \Lambda^2{}_2 \Lambda^1{}_0 F^{20} + \Lambda^2{}_2 \Lambda^1{}_1 F^{21}$$

$$= -\gamma v E_y + \gamma B_z$$

$$\begin{cases} E'_x = E_x \\ E'_y = \gamma(E_y - v B_z) \\ E'_z = \gamma(E_z + v B_y) \end{cases} \qquad \begin{cases} B'_x = B_x \\ B'_y = \gamma(B_y + v E_z) \\ B'_z = \gamma(B_z - v E_y) \end{cases}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

$$\tilde{F}^{01} = \frac{1}{2}\left(\varepsilon^{0123} F_{23} + \varepsilon^{0132} F_{32}\right) = F_{23} = -B_x$$

$$\tilde{F}^{02} = -B_y$$

$$\tilde{F}^{03} = -B_z$$

$$\tilde{F}^{12} = \frac{1}{2}\left(\varepsilon^{1230} F_{30} + \varepsilon^{1203} F_{03}\right) = +E_z$$

$$\varepsilon^{1230} = -1$$

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & +E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \qquad F_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}$$

4) $F^{\mu\nu} F_{\mu\nu} = 2(\vec{B}^2 - \vec{E}^2)$

$F^{\mu\nu} \tilde{F}_{\mu\nu} = -4\,\vec{E}\cdot\vec{B}$

$\tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} = -2(B^2 - E^2)$

Scalaire sous Lorentz.

Pseudo-Scalaire

5) $\partial_\mu F^{\mu\nu}$?

$$\partial_\mu F^{\mu 0} = \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \underbrace{\vec{\nabla}\cdot\vec{E}}_{\text{Maxwell}} = \rho = J^0 \quad \text{Ok}$$

$$\partial_\mu F^{\mu 1} = \partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31}$$

$$= -\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}$$

$$= \underbrace{\left(\vec{\nabla}\times\vec{B} - \frac{\partial \vec{E}}{\partial t}\right)\Big|_x}_{\text{Maxwell}} = J_x = J^1 \quad \text{Ok}$$

etc...

$$\partial_\mu \tilde{F}^{\mu 0} = \partial_1 \tilde{F}^{10} + \partial_2 \tilde{F}^{20} + \partial_3 \tilde{F}^{30}$$
$$= \underbrace{\vec{\nabla}\cdot\vec{B} = 0}_{\text{Maxwell}} \qquad \text{OK.}$$

$$\partial_\mu \tilde{F}^{\mu 1} = \partial_0 \tilde{F}^{01} + \partial_2 \tilde{F}^{21} + \partial_3 \tilde{F}^{31}$$
$$= -\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}$$
$$= \underbrace{\left(-\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B}\right)\Big|_x = 0}_{\text{Maxwell.}} \qquad \text{OK}$$

$$\partial_\mu J^\mu = \underbrace{\partial_\mu \partial_\nu}_{\text{Symétrique}} \underbrace{F^{\mu\nu}}_{\text{Antisymétrique}}$$

$= 0.$

$$\frac{\partial \rho}{\partial t} - \vec{\nabla}\cdot\vec{J} = 0 \Rightarrow \underbrace{\int_V \frac{\partial \rho}{\partial t}}_{\text{Variation de la charge dans le volume } V} = \int_S \vec{J}\cdot d\vec{S}$$

$\int_S \vec{J}\cdot d\vec{S}$ = courant à travers surface ...

↳ conservation locale de la charge électrique.

6] $$\frac{\delta S}{\delta A^\rho(x')} = \frac{\delta}{\delta A^\rho(x')}\left\{-\int d^4x \left(\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu + J_\mu A^\mu\right)\right.$$
$$= -\int d^4x \left(\frac{1}{2}\partial_\mu \delta^{(4)}(x-x')\partial^\mu A_\rho + \frac{1}{2}\partial_\mu A_\rho \partial^\mu \delta^{(4)}(x-x')\right.$$
$$\left.\left. -\frac{1}{2}\partial_\mu \delta^{(4)}(x-x')\partial_\rho A^\mu - \frac{1}{2}\partial_\rho A_\mu \partial^\mu \delta^{(4)}(x-x') + J_\rho \delta^{(4)}(x-x')\right)\right\}$$

$$= - \left(-\frac{2}{2} \partial_\mu \partial^\mu A_\rho + \frac{2}{2} \partial_\mu \partial_\rho A^\mu + J_\rho \right).$$

$$= - \left(\partial_\mu F_\rho{}^\mu + J_\rho \right) = 0$$

ce qui redonne bien : $\partial_\mu F^{\mu\nu} = J^\nu$.

Si $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$\partial_\mu \tilde{F}^{\mu\nu} = \frac{1}{2} \partial_\mu \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

$$= \frac{1}{2} \partial_\mu \varepsilon^{\mu\nu\rho\sigma} (\partial_\rho A_\sigma - \partial_\sigma A_\rho)$$

$$= \underbrace{\varepsilon^{\mu\nu\rho\sigma}}_{\text{impair}} \underbrace{\partial_\mu \partial_\rho}_{\text{pair}} A_\sigma = 0.$$

7] $F^{\mu\nu}\tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$

$$S_2[A] = \lambda \int d^4x \; \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

$$\frac{\delta S_2}{\delta A^\tau(x')} = 2\lambda \int d^4x \; \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} \frac{\delta F^{\rho\sigma}}{\delta A^\tau(x')}$$

$$= 2\lambda \int d^4x \; \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} \left[\partial^\rho \delta^{(4)}(x-x') \, \delta^\sigma_\tau - \partial^\sigma \delta^{(4)}(x-x') \, \delta^\rho_\tau \right]$$

$$= -2\lambda \left[\varepsilon_{\mu\nu\rho\tau} \partial^\rho F^{\mu\nu} - \varepsilon_{\mu\nu\tau\sigma} \partial^\sigma F^{\mu\nu} \right].$$

$$= 4\lambda \, \varepsilon_{\mu\nu\tau\sigma} \partial^\sigma F^{\mu\nu} = \cancel{-8\lambda \partial_\mu \tilde{F}^{\mu\nu}}$$

$$= -8\lambda \, \partial^\mu \tilde{F}_{\mu\tau} = 0$$

En fait, $S_2[A] = 4\lambda \int d^4x \; \varepsilon_{\mu\nu\rho\sigma} (\partial^\mu A^\nu)(\partial^\rho A^\sigma)$.

$$= 4\lambda \int d^4x \; \partial^\mu \left(\varepsilon_{\mu\nu\rho\sigma} A^\nu \partial^\rho A^\sigma \right)$$

$$= 4\lambda \oint_{\text{surface}} \varepsilon_{\mu\nu\rho\sigma} A^\nu \partial^\rho A^\sigma \, dS^\mu$$

↳ la dérivée fonctionnelle dans le bulk est nulle.