**M2 ICFP**
**Theoretical Condensed Matter**

*B. Douçot, B.Estienne, L. Messio*

# Landau theory of Fermi liquids

We start from a Fermi gas (without interactions, namely $H = \sum_{\mathbf{p},\sigma} \epsilon^0_{\mathbf{p}} c^\dagger_{\mathbf{p},\sigma} c_{\mathbf{p},\sigma}$) at zero temperature and add interactions to get a Fermi liquid. We suppose the system is isotropic and set $\hbar = 1$. In the gas, infinitely long lived particle-hole excitations are constructed from the one-particle spectrum $\epsilon_{\mathbf{p}}$. The ground state is characterized by the distribution function $n^0_{\mathbf{p}} = \theta(p_F - p)$.

*Landau Fermi liquid theory* is an effective theory describing the low-energy degrees of freedom of a Fermi gas with interactions. The main assumption is that the low-energy excited states are adiabatically connected to the non interacting ones. Then the Fermi liquid excitations are labelled by the same occupation numbers $n_{\mathbf{p}}$ as the non-interacting ones. The main difference with the non interacting gas is that these elementary excitations interact with each other. As a result they acquire a finite life time, and for this reason these excitations are called *quasiparticles* (and *quasiholes*). Fortunately these quasiparticles are better and better defined as one approaches the Fermi surface[1], and as long as we consider only low energy excitations (*i.e.* close to the FS), the quasiparticle damping can be neglected.

In the interacting system, $n_{\mathbf{p}}$ describes the distribution of quasiparticles, and is measured by the departure from the ground state distribution $\delta n_{\mathbf{p}} = n_{\mathbf{p}} - n^0_{\mathbf{p}}$. We will only consider low energy excitations for which $\delta n_{\mathbf{p}}$ is non zero only for $\mathbf{p}$ close to the FS.

## 1 Introduction to Landau Fermi liquids

1. We consider the state obtained from a perturbation involving a small displacement $\delta p$ of the Fermi surface. What is the sign of $\delta n_{\mathbf{p}}$ if $\mathbf{p}$ is
   - deep in or far out of the Fermi sphere ?
   - is newly in the FS ?
   - is newly out of the FS ?
2. Near $|\mathbf{p}| = p_F$, the Fermi velocity is given by $\epsilon_{\mathbf{p}} - \mu \sim v_F(p - p_F)$, and the effective mass by $m^* = p_F/v_F$. Recover the energy density of particle states at the Fermi surface, for a periodic system of size $L$ and of volume $\Omega = L^3$ :
$$N^0 = \frac{\Omega\, m^*\, p_F}{\pi^2}.$$
3. The excitation energy at zero temperature can be developed in $\delta n_{\mathbf{p}}$. A naive expansion of $E$ to order $\delta n_{\mathbf{p}}$ is:
$$E = E_0 + \sum_{\mathbf{p}} (\epsilon_{\mathbf{p}} - \mu)\delta n_{\mathbf{p}} \qquad (1)$$
   where $E_0$ is the ground state energy.

   For a displacement of the Fermi surface by a small amount $\delta p$, what is the order of $\epsilon_{\mathbf{p}} - \mu$ and $\sum_{\mathbf{p}} (\epsilon_{\mathbf{p}} - \mu)\delta n_{\mathbf{p}}$ in $\delta p$ ? Is it reasonable to neglect the quadratic terms in $\delta n_{\mathbf{p}}$ in the expansion (1) ?

[1] At zero temperature, this lifetime varies as the inverse square of the energy separation to the Fermi surface (*cf* calculation of the self-energy).

4. The phenomenological theory of Fermi liquids as proposed by Landau takes into account interaction between quasiparticles through an extra quadratic term

$$E = E_0 + \sum_{\mathbf{p}} (\epsilon_{\mathbf{p}} - \mu)\delta n_{\mathbf{p}} + \frac{1}{2}\sum_{\mathbf{pp'}} f_{\mathbf{pp'}} \delta n_{\mathbf{p}} \delta n_{\mathbf{p'}} \tag{2}$$

where $f_{\mathbf{pp'}}$ are the Landau parameters. They are symmetric.

What is the order of $f_{\mathbf{pp'}}$ in the volume $\Omega$ ? Give a physical justification.

5. What is the energy $\bar{\epsilon}_{\mathbf{p}}$ of an additional quasiparticle with momentum $\mathbf{p}$?

We now introduce the spin of particles ($\sigma = \pm 1/2$) and suppose that the system is time reversal invariant (no magnetic field):

$$E = E_0 + \sum_{\mathbf{p}\sigma} (\epsilon_{\mathbf{p}} - \mu)\delta n_{\mathbf{p}\sigma} + \frac{1}{2}\sum_{\mathbf{pp'}\sigma\sigma'} f^{\sigma\sigma'}_{\mathbf{pp'}} \delta n_{\mathbf{p}\sigma} \delta n_{\mathbf{p'}\sigma'}. \tag{3}$$

Because of the symmetries, we can split the Landau parameters into symmetric and antisymmetric coefficients:

$$f^{\sigma\sigma}_{\mathbf{pp'}} = f^{s}_{\mathbf{pp'}} + f^{a}_{\mathbf{pp'}}$$
$$f^{\sigma-\sigma}_{\mathbf{pp'}} = f^{s}_{\mathbf{pp'}} - f^{a}_{\mathbf{pp'}}.$$

As only wave vectors near the Fermi surface are considered in our isotropic system, only the relative angle $\theta$ between $\mathbf{p}$ and $\mathbf{p'}$ is important. We can expand the coefficients in term of Legendre polynomials:

$$f^{s(a)}_{\mathbf{pp'}} = \sum_{l=0}^{\infty} f^{s(a)}_{l} P_l(\cos\theta). \tag{4}$$

We recall the orthogonality relation: $\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{mn}$ and give $P_0(x) = 1$ and $P_1(x) = x$. We also define $F^{s(a)}_l = N^0 f^{s(a)}_l$.

Implicit in Landau theory is the hope that the series Eq. (4) converges rapidly with increasing $l$. As we shall now see, leading parameters can be related to experimentally measurable quantities: effective mass, specific heat, spin and charge susceptibility and first sound velocity.

## 2 Effective mass

We consider the system from a moving frame at an infinitesimal velocity $\mathbf{v}$ with respect to the laboratory frame.

6. What is the Hamiltonian $H'$ in the moving frame as a function of $\mathbf{v}$, of the initial Hamiltonian $H$, of the total mass $M$ and of the total momentum $\mathbf{P}$ of the system ?

7. We take as our reference (excited) state the ground state of $H$ viewed from the moving frame. Show that in the thermodynamical limit:

$$\delta n_{\mathbf{p}\sigma} = -\frac{m}{m^*}\mathbf{p}\cdot\mathbf{v}\,\delta(\epsilon_{\mathbf{p}} - \epsilon_F).$$

8. We add a quasiparticle of momentum $\mathbf{p}$ (in the moving frame) and of spin $\sigma$ to the system. Calculate its energy $\epsilon'_{\mathbf{p}\sigma}$ in the moving frame as a function of $\epsilon_{\mathbf{p}\sigma}$, $\mathbf{v}\cdot\mathbf{p}$, $m$ and $m^*$. Now, calculate it from the energy of Q.5.

$$\boxed{\frac{m^*}{m} = 1 + \frac{1}{3}F^s_1} \tag{5}$$

## 3 Magnetic susceptibility

We now determine the spin susceptibility $\chi$ of a Fermi liquid. $\chi = \frac{1}{\Omega}\frac{dM}{dH}\big|_{H\to 0}$, where $H$ is the external applied magnetic field in the $z$ direction. The Zeeman coupling causes a change of energy of $-\gamma\sigma H$, where $\gamma$ is the gyromagnetic ratio. Our reference state is the equilibrium state of the Fermi liquid under $H$.

9. Is the chemical potential $\mu$ affected by $H$ to first order in $H$ (for a constant number of particles) ? Why ? Relate $\delta n_{\mathbf{p},\sigma}$ to $\delta n_{\mathbf{p},-\sigma}$.
10. What is the energy $\bar{\epsilon}_{\mathbf{p}\sigma}$ of a quasiparticle near the Fermi surface ? Express it as a function of $\Delta n_\sigma = \sum_{\mathbf{p}} \delta n_{\mathbf{p}\sigma}$.
11. Calculate $\Delta n_\sigma$ and deduce that

$$\boxed{\chi = \frac{\gamma^2 m^* p_F}{4\pi^2(1+F_0^a)}} \tag{6}$$

Other quantities such as the compressibility ($\kappa = \frac{1}{\rho^2}\frac{\partial\rho}{\partial\mu}$) and the specific heat ($C_v = \frac{\partial E}{\partial T}\big|_N$) can be calculated. The derivation can be found in the book from Pines and Nozières.

## 4 Compressibility

12. The compressibility is given by $\kappa = \frac{1}{\rho^2}\frac{\partial\rho}{\partial\mu}$ where $\rho$ is the particule density. Show that

$$\kappa = \frac{p_F m^*}{\rho^2\pi^2(1+F_0^s)} \tag{7}$$