M2 ICFP
Theoretical Condensed Matter

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Problem: Landau theory of Fermi liquids

We start from a system of non-interacting electrons (a *Fermi gas*) at zero temperature, with Hamiltonian

$$H = \sum_{\mathbf{p},\sigma} \epsilon^0_{\mathbf{p}} c^\dagger_{\mathbf{p},\sigma} c_{\mathbf{p},\sigma}$$

with $\epsilon^0_{\mathbf{p}} = p^2/2m$ and $\sigma = \pm 1/2$. We suppose that the system is isotropic and set $\hbar = 1$. In the Fermi gas, infinitely long lived particle-hole excitations are constructed from the one-particle spectrum $\epsilon^0_{\mathbf{p}}$. The ground state is characterized by the Fermi-Dirac step function $n^0_{\mathbf{p},\sigma} = \theta(p_F - p)$, and excited states by $\delta n_{\mathbf{p},\sigma} = n_{\mathbf{p},\sigma} - n^0_{\mathbf{p},\sigma}$. For instance a particle excitation of momentum $\mathbf{p}'$ ($p' > p_F$) and spin σ' corresponds to $\delta n_{\mathbf{p},\sigma} = \delta_{\mathbf{p},\mathbf{p}'}\delta_{\sigma,\sigma'}$, while a hole excitation of momentum $\mathbf{p}'$ ($p' < p_F$) and spin σ' is described by $\delta n_{\mathbf{p},\sigma} = -\delta_{\mathbf{p},\mathbf{p}'}\delta_{\sigma,\sigma'}$.

1. For a periodic system of N non-interacting electrons of mass m in a large volume V, what is the Fermi surface? Compute the Fermi momentum p_F and Fermi energy ϵ^0_F. What is the energy of a generic excited state $\delta n_{\mathbf{p},\sigma}$? Show that the chemical potential is equal to the energy of a particle *on the Fermi surface*.

2. What is the density of state $\nu^0(\epsilon)$?

3. We set $k_B = 1$. Show that the entropy S of the Fermi gas in the state $n_{\mathbf{p},\sigma}$ is

$$S[n] = -\sum_{\mathbf{p},\sigma} \left[n_{\mathbf{p},\sigma} \ln n_{\mathbf{p},\sigma} + (1 - n_{\mathbf{p},\sigma}) \ln(1 - n_{\mathbf{p},\sigma}) \right] \tag{1}$$

4. The thermodynamic potential is given by

$$\Omega[n] = E[n] - \mu N[n] - TS[n] \tag{2}$$

where $E[n] = \sum_{\mathbf{p},\sigma} \epsilon^0_{\mathbf{p}} n_{\mathbf{p},\sigma}$ and $N[n] = \sum_{\mathbf{p},\sigma} n_{\mathbf{p},\sigma}$. The equilibrium distribution n^{eq} is obtained as $\delta\Omega[n]/\delta n_{\mathbf{p},\sigma} = 0$. Recover the Fermi-Dirac distribution.

1 The quasiparticle concept

Upon adding interactions, the property of a Fermi gas can change drastically (phase transition, for example to a superconducting state) but in a *normal* Fermi liquid, many properties of the non-interacting gas are unchanged. *Landau Fermi liquid theory* is an effective theory describing the low-energy properties of *normal* Fermi liquids. The main idea underlying this construction is *adiabaticity*: slowly switching on the interactions, Landau argued that the ground state adiabatically transforms into the ground state of the interacting system. During this adiabatic process, conserved quantities such as spin, charge or momentum remain unchanged, while dynamical properties such as mass, magnetic moment, *etc* are renormalized to new values.

With interactions, a non-interacting excited state $\delta n_{\mathbf{p},\sigma}$ gets dressed and becomes an eigenstate of the interacting system. Excitations of the Fermi liquid are no more particule or hole excitations, but are fully interacting dressed states called *quasi-particles* or *quasi-holes*, with a finite lifetime. Moreover, they now interact with each other. In the interacting system, $n_{\mathbf{p},\sigma}$ describes the distribution of quasiparticles, and is measured by the departure from the ground state distribution $\delta n_{\mathbf{p},\sigma} = n_{\mathbf{p},\sigma} - n^0_{\mathbf{p},\sigma}$. We will only consider low energy excitations for which $\delta n_{\mathbf{p},\sigma}$ is small, and non zero only for $\mathbf{p}$ close to the FS. In this regime the energy can be developed in $\delta n_{\mathbf{p},\sigma}$:

$$E[n] = E_0 + \sum_{\mathbf{p},\sigma} \epsilon_{\mathbf{p}} \delta n_{\mathbf{p},\sigma} + \frac{1}{2V} \sum_{\mathbf{p},\mathbf{p}',\sigma,\sigma'} f^{\sigma\,\sigma'}_{\mathbf{p}\,\mathbf{p}'} \delta n_{\mathbf{p},\sigma} \delta n_{\mathbf{p}',\sigma'} + O(\delta n^3), \tag{3}$$

where $f^{\sigma\,\sigma'}_{\mathbf{p}\,\mathbf{p}'}$ are the Landau parameters.

The quasiparticle dispersion relation $\epsilon_{\mathbf{p}}$ can be expanded around the Fermi surface as

$$\epsilon_{\mathbf{p}} \sim \epsilon_F + v^*_F(|\mathbf{p}| - p_F), \qquad v^*_F = \frac{p_F}{m^*} \tag{4}$$

which defines the (renormalized) Fermi velocity v^*_F and effective mass m^*. By analogy with the non-interacting case, we define the quasiparticle density at the Fermi surface as

$$\nu^*(\epsilon_F) = \frac{V\, m^*\, p_F}{\pi^2}.$$

5. What describes the quadratic term of equation (3)?
6. If V is the total volume, what is the order of $f^{\sigma\,\sigma'}_{\mathbf{p}\,\mathbf{p}'}$ in V? Give a physical justification.
7. Suppose that $\delta n_{\mathbf{p},\sigma}$ is only significant for $|p - p_F| < \delta$. Show that both the linear term and the quadratic term are of the same order in δ.
8. What is the energy $\tilde{\epsilon}_{\mathbf{p}}$ of an additional quasiparticle with momentum $\mathbf{p}$ in the excited state $\delta n_{\mathbf{p},\sigma}$?

We can fix $f^{\sigma\,\sigma'}_{\mathbf{p}\,\mathbf{p}'} = f^{\sigma'\,\sigma}_{\mathbf{p}'\,\mathbf{p}}$. For a time-reversal invariant system, we can split the Landau parameters into symmetric and antisymmetric coefficients:

$$\begin{cases} f^{\sigma\sigma}_{\mathbf{p}\mathbf{p}'} = f^{s}_{\mathbf{p}\mathbf{p}'} + f^{a}_{\mathbf{p}\mathbf{p}'} \\ f^{\sigma-\sigma}_{\mathbf{p}\mathbf{p}'} = f^{s}_{\mathbf{p}\mathbf{p}'} - f^{a}_{\mathbf{p}\mathbf{p}'}. \end{cases}$$

As only wave vectors near the Fermi surface are considered in our isotropic system, we can set $|\mathbf{p}| = |\mathbf{p}'| = p_F$, and only the relative angle θ between $\mathbf{p}$ and $\mathbf{p}'$ is important. $\hat{\mathbf{p}}$ denotes a unit vector in the direction of $\mathbf{p}$. We can expand the coefficients in term of Legendre polynomials:

$$f^{s(a)}_{\mathbf{p}\mathbf{p}'} = \sum_{l=0}^{\infty} f^{s(a)}_l P_l(\cos\theta), \qquad f^{s(a)}_l = (2l+1) \int \frac{d\hat{\mathbf{p}}'}{4\pi} f^{s(a)}_{\mathbf{p}\mathbf{p}'} P_l(\cos\theta) \tag{5}$$

We recall that $P_0(x) = 1$ and $P_1(x) = x$ and the orthogonality relation:

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{mn}.$$

We introduce the dimensionless parameters

$$F^{s(a)}_l = \frac{1}{V} \nu^*(\epsilon_F) f^{s(a)}_l = \frac{m^*\, p_F}{\pi^2} f^{s(a)}_l$$

Now, we will determine the properties of Fermi liquids and compare them with those of the Fermi gas.

2 Effective mass

We consider the system from a moving frame at an infinitesimal velocity $\mathbf{v}$ with respect to the laboratory frame.

9. Let $|\Psi\rangle$ be an eigenstate of H with energy E and total momentum $\mathbf{P}$. What is the energy E' in the moving frame as a function of $\mathbf{v}$, of the initial energy E, of the total mass M and of the total momentum $\mathbf{P}$? * Figure out how to implement Galilean transformations using a (time-dependent) unitary transformation.

10. We take as our reference (excited) state the ground state of H viewed from the moving frame. Show that
$$\delta n_{\mathbf{p}} = -\frac{m}{m^*}\mathbf{p}\cdot\mathbf{v}\delta(\epsilon_{\mathbf{p}} - \epsilon_F).$$

11. We add a quasiparticle of momentum $\mathbf{p}$ (in the moving frame) to the system. Calculate its energy $\epsilon'_{\mathbf{p}}$ in the moving frame firstly as a function of $\epsilon_{\mathbf{p}}$, $\mathbf{v}\cdot\mathbf{p}$, m and m^* and secondly from the energy of Q.8. Show that:
$$\boxed{\frac{m^*}{m} = 1 + \frac{1}{3}F_1^s}. \tag{6}$$

3 Magnetic susceptibility

We now determine the spin susceptibility χ of a Fermi liquid. $\chi = \frac{1}{V}\left.\frac{dM}{dB}\right|_{B\to 0}$, where B is the external magnetic field in the z direction. The Zeeman coupling causes an energy change for a particle of $-\gamma\sigma B$, where γ is the gyromagnetic ratio. Our reference state is the equilibrium state of the Fermi liquid without magnetic field and we note $\delta n_{\mathbf{p},\sigma}$ the difference of occupation in the presence of B.

12. Is the chemical potential μ affected by B to first order in B (for a constant number of particles)? Why? Relate $\delta n_{\mathbf{p},\sigma}$ to $\delta n_{\mathbf{p},-\sigma}$.

13. What is the energy $\tilde{\epsilon}_{\mathbf{p}\sigma}$ of a quasiparticle near the Fermi surface for a state under a magnetic field B? Express it as a function of $\Delta n_\sigma = \sum_{\mathbf{p}} \delta n_{\mathbf{p}\sigma}$.

14. Calculate Δn_σ and deduce that
$$\boxed{\chi = \frac{\gamma^2\nu^*(\epsilon_F)}{4V(1+F_0^a)}} \tag{7}$$

4 Compressibility

15. The compressibility is given by $\kappa = \frac{1}{\rho^2}\frac{\partial\rho}{\partial\mu}$ where ρ is the particule density N/V. Show that
$$\kappa = \frac{p_F m^*}{\rho^2\pi^2(1+F_0^s)}. \tag{8}$$

As an intermediate step, you can calculate the energy of a quasiparticle at the new Fermi energy.

5 Stability of the ground state and Pomeranchuk instabilities

We consider an excited state where all state inside some spin dependent surface near the Fermi surface are filled. In the $\mathbf{p}$ direction, the Fermi surface is displaced by $u_\sigma(\hat{\mathbf{p}})$.

(a) Calculate the difference of free energy at zero temperature for infinitesimal $u_\sigma(\hat{\mathbf{p}})$ and put it under the form:

$$\Delta(E-\mu N) = \frac{\nu^*(\epsilon_F)v_F^2}{4}\sum_{\sigma,\sigma'}\left(\delta_{\sigma,\sigma'}\int\frac{d\Omega_p}{4\pi}u_\sigma(\hat{p})^2 + \frac{1}{2}\int\frac{d\Omega_p}{4\pi}\int\frac{d\Omega_{p'}}{4\pi}F^{\sigma\,\sigma'}_{\hat{p},\hat{p}'}u_\sigma(\hat{p})u_{\sigma'}(\hat{p}')\right) \tag{9}$$

Using the expansion in spherical harmonics:

$$u_\sigma(\hat{\mathbf{p}}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(u^s_{lm}+\sigma u^a_{lm})Y_l^m(\hat{\mathbf{p}}), \tag{10}$$

and the fact that for any u, the difference of free energy must be positive to insure the stability of the ground state, we get the Pomeranchuk inequalities:

$$F_l^s > -2l-1, \qquad F_l^a > -2l-1. \tag{11}$$