M2 ICFP
Theoretical Condensed Matter

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Problem: The Gross-Neveu model, symmetries, mean-field approximation and Goldstone modes

The subject of the problem is the famous Gross-Neveu model. It has been studied a lot in the 1970's in the high energy physics community, as a theoretical laboratory to investigate non-perturbative phenomena in strongly interacting quantum field theories. From a solid-state physics standpoint, it appears as a rather natural generalization of the Luttinger model with spin, where the global symmetry group $SU(2)$ is replaced by $SU(N)$, N arbitrary positive integer. A lot of attention has been dedicated to the large N limit, for which a rather appealing physical picture has been proposed by E. Witten in 1978, using an approach based on path-integrals. The following problem aims at presenting this physics from the viewpoint and methods used in the course.

Before moving on to the Gross-Neveu model proper, we are going to review two technical points related to second quantization in the continuum : fermion doubling, and normal ordering.

1 Fermion doubling

Consider a non-interacting, one-dimensional tight-binding model on N sites with periodic-boundary conditions[1].

$$H = -t \sum_{j=1}^{N} \left(c_{j+1}^{\dagger} c_j + \text{h.c.} \right), \qquad c_{N+j} = c_j$$

As we have seen in the tutorial about the Hubbard model, such a quadratic model is straightforward to solve. We have

$$H = \sum_k \epsilon_k \tilde{c}_k^{\dagger} \tilde{c}_k, \qquad \epsilon_k = -2t \cos k$$

where

$$\tilde{c}_k^{\dagger} = \frac{1}{\sqrt{N}} \sum_j e^{ikj} c_j^{\dagger}, \qquad k = \frac{2\pi}{N} m, \quad m = 0, 1, \cdots, N-1$$

At half-filling, all the states with a negative energy are filled.

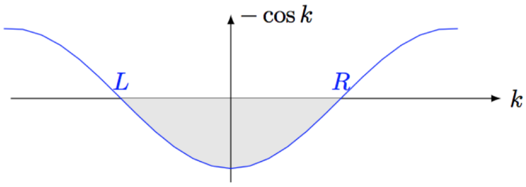
- To simplify our analysis we will assume that the number of sites N is even. What is the ground state ? Show that for $N = 0$ mod 4, the ground state is four-fold degenerate. This is due to the presence of so-called *zero modes* (modes with an energy 0). Show that when $N = 2$ mod 4, the ground state is unique. Argue that the ground-state energy diverges when $N \to \infty$.

[1] The other natural boundary condition for fermions is anti-periodic : $c_{N+j} = -c_j$

We are concerned with the thermodynamic limit of this simple model of one-dimensional fermion. Before taking the thermodynamic limit, we first re-introduce the lattice spacing a, so that sites are located at positions $x = ja$, and the total chain length is $L = Na$. The thermodynamic limit is obtained by taking $a \to 0$, $N \to \infty$, keeping x and L constant.

The low energy/ long distance physics is dominated by the momenta close to the Fermi surface. The relevant momenta values are $k = \pm\frac{\pi}{2} + aq$, with q not to large[2]. It is then natural to drop the fast moving degrees of freedom and only keep the low-energy terms in the Fourier expansion of the fermion operator

$$c_j = \frac{1}{\sqrt{N}} \sum_k \tilde{c}_k e^{ikj} \to \sqrt{\frac{a}{L}} \left(e^{i\frac{\pi}{2}j} \sum_q c_{\frac{\pi}{2}+aq} e^{iqx} + e^{-i\frac{\pi}{2}j} \sum_q c_{-\frac{\pi}{2}+aq} e^{iqx} \right), \qquad \text{where } x = ja$$



We get a left moving fermion around $k = -\frac{\pi}{2}$, and a right moving one around $k = \frac{\pi}{2}$. They are called left (resp. right) moving because their (linearized) dispersion relation is $\epsilon(k) = -v_F k$ (resp $\epsilon(k) = v_F k$).

$$\Psi_R(x) = \sqrt{\frac{1}{L}} \sum_q \underbrace{\tilde{c}_{\frac{\pi}{2}+aq}}_{c_R(q)} e^{iqx}, \qquad \Psi_L(x) = \sqrt{\frac{1}{L}} \sum_q \underbrace{\tilde{c}_{-\frac{\pi}{2}+aq}}_{c_L(q)} e^{iqx}$$

we get

$$\boxed{c_j = \sqrt{a} \left(e^{-i\pi j/2} \Psi_L(x) + e^{i\pi j/2} \Psi_R(x) \right), \qquad x = ja}$$

- Argue that in the limit $a \to 0$ the two fermions Ψ_L and Ψ_R become independent : the lattice fermion operator c_j yields two fermion fields in the continuum ! This phenomenon is due to the fact that there are two momentum region in the low-energy limit. Are the fermions Ψ_η periodic ? Show that we have

$$\boxed{\{\Psi^\dagger_\eta(x), \Psi_{\eta'}(x')\} = \delta_{\eta,\eta'} \delta(x - x')}$$

- Show that in the continuum limit, the non-interacting fermionic Hamiltonian becomes

$$\begin{aligned} H &= v_F \int_0^L dx\, i \left(\Psi^\dagger_L(x) \partial_x \Psi_L(x) - \Psi^\dagger_R(x) \partial_x \Psi_R(x) \right) \\ &= v_F \sum_q q \left(c^\dagger_R(q) c_R(q) - c^\dagger_L(q)_L(q) \right) \end{aligned}$$

What is the Fermi velocity v_F ?

[2]Not too large meaning that we stay at low energy, in the regime where the dispersion relation can be linearized.

2 Normal order

- Let us denote by $|\mathrm{GS}\rangle$ the ground state (Fermi see) of the continuum model, which we are going to call the *vacuum*. Show that for $q > 0$ we have

$$c_R^\dagger(-q)|\mathrm{GS}\rangle = 0 \qquad \text{and} \qquad c_R(q)|\mathrm{GS}\rangle = 0$$
$$c_L^\dagger(q)|\mathrm{GS}\rangle = 0 \qquad \text{and} \qquad c_L(-q)|\mathrm{GS}\rangle = 0$$

Comment on the energy of the ground-state.

The ground-state energy problem is typical of a field theory, and is due to the infinite number of degrees of freedom. It is however very easy to circumvent. Before taking the continuum limit, we can subtract the GS energy from the Hamiltonian, which amounts to put the GS energy to 0. The Hamiltonian in the continuum becomes

$$\boxed{H_F = v_F \sum_q q \left(: c_R^\dagger(q) c_R(q) : - : c_L^\dagger(q)_L(q) : \right)}$$

where the normal ordered product $: c_\eta^\dagger(k) c_\eta(k) :$ is defined as

$$: c_\eta^\dagger(k) c_\eta(k) := c_\eta^\dagger(k) c_\eta(k) - \langle \mathrm{GS}| c_\eta^\dagger(k) c_\eta(k) |\mathrm{GS}\rangle$$

- Show that

$$: c_L^\dagger(k) c_L(k) := \begin{cases} -c_L(k) c_L^\dagger(k) & \text{if} \quad k > 0 \\ c_L^\dagger(k) c_L(k) & \text{if} \quad k < 0 \end{cases}, \qquad : c_R^\dagger(k) c_R(k) := \begin{cases} c_R^\dagger(k) c_R(k) & \text{if} \quad k > 0 \\ -c_R(k) c_R^\dagger(k) & \text{if} \quad k < 0 \end{cases}$$

So far we have defined the normal order of quadratic terms $: c_\alpha^\dagger c_\beta :$. For more general expression, normal order is defined by moving all the c and $c^\dagger$ that annihilate the ground state to the right, with a global sign that account for how many c and $c^\dagger$ have been permuted in the process. For instance, assuming $q_1 < 0$, $q_2 > 0$, $q_1 + k > 0$, $q_2 - k < 0$, we have

$$: c_R^\dagger(q_2 - k) c_R^\dagger(q_1 + k) c_R(q_2) c_R(q_1) := -c_R^\dagger(q_1 + k) c_R(q_1) c_R^\dagger(q_2 - k) c_R(q_1)$$

because between these four operators, only $c_R(q_1)$ and $c_R^\dagger(q_2 - k)$ annihilate the GS. The relative order of the operators that annihilate the GS can be chosen freely (it just changes the sign). Likewise for their hermitian conjugates. So we could just as well write

$$\begin{aligned} : c_R^\dagger(q_2 - k) c_R^\dagger(q_1 + k) c_R(q_2) c_R(q_1) : &= c_R(q_1) c_R^\dagger(q_1 + k) c_R^\dagger(q_2 - k) c_R(q_1) \\ &= -c_R(q_1) c_R^\dagger(q_1 + k) c_R(q_1) c_R^\dagger(q_2 - k) \end{aligned}$$

3 Gross-Neveu model

The Gross-Neveu Hamiltonian reads:

$$\begin{aligned} H_{\mathrm{GN}} &= \sum_k \sum_{\sigma=1}^N k(: c_{R\sigma}^\dagger(k) c_{R\sigma}(k) : - : c_{L\sigma}^\dagger(k) c_{L\sigma}(k) :) \\ &\quad + N g_1 \int_0^L dx \ : \mathcal{O}(x) \mathcal{O}^\dagger(x) : + \frac{g_2}{N} \int_0^L dx \ : n_R(x) n_L(x) : \end{aligned} \tag{1}$$

We use units in which $\hbar = 1$. The system in one-dimensional with length L, and periodic boundary conditions. The symbol :: stands for normal-ordering, R for right-moving fermions and L for left-moving fermions. The generalized spin index σ runs now from 1 to N. The operator $\mathcal{O}(x)$ is defined by:

$$\mathcal{O}(x) = \frac{1}{N}\sum_{\sigma=1}^{N} \Psi^\dagger_{R\sigma}(x)\Psi_{L\sigma}(x). \tag{2}$$

The local charge densities $n_R(x)$ and $n_L(x)$ and the total charges N_R and N_L are:

$$n_\alpha(x) = \sum_{\sigma=1}^{N} \Psi^\dagger_{\alpha\sigma}(x)\Psi_{\alpha\sigma}(x), \qquad N_\alpha = \int_0^L dx\ :n_\alpha(x): \tag{3}$$

In the sequel, it will be often made use of

$$J = N_R - N_L.$$

The conventions for the Fourier transforms of the fields $\Psi_\eta(x)$ are fixed to:

$$\Psi_\eta(x) = \frac{1}{\sqrt{L}}\sum_k c_\eta(k)e^{ikx}, \qquad c_\eta(k) = \frac{1}{\sqrt{L}}\int_0^L dx\, e^{-ikx}\Psi_\eta(x). \tag{4}$$

1. What is the Fermi surface of the free model ? What is the Fermi velocity ?
2. The Luttinger model neglects some terms in the local density. Show that taking these terms into account give the Gross-Neveu Hamiltonian.

4 Symmetries

3. What are the commutators $[J, \mathcal{O}(x)]$ and $[J, \mathcal{O}^\dagger(x)]$?
4. Check that J commutes with H_{GN}. The associated symmetry is often called *chiral symmetry* in the high energy physics literature.
5. It is interesting to see how this symmetry J acts on basic operators. Let us introduce

   $$U(\phi) = e^{i\frac{\phi}{2}J}.$$

   Evaluate then $U(\phi)\Psi^\dagger_{R\sigma}(x)U(\phi)^{-1}$, $U(\phi)\Psi^\dagger_{L\sigma}(x)U(\phi)^{-1}$, $U(\phi)n_R(x)U(\phi)^{-1}$, $U(\phi)n_L(x)U(\phi)^{-1}$, $U(\phi)\mathcal{O}(x)U(\phi)^{-1}$, and $U(\phi)\mathcal{O}^\dagger(x)U(\phi)^{-1}$.
6. Another symmetry of H_{GN} is the particle-hole symmetry. It is implemented by the linear operator $\mathcal{C}$ defined by:

   $$\mathcal{C}^2 = 1, \qquad \mathcal{C}\Psi^\dagger_{R\sigma}(x)\mathcal{C} = \Psi_{R\sigma}(x), \qquad \mathcal{C}\Psi^\dagger_{L\sigma}(x)\mathcal{C} = -\Psi_{L\sigma}(x). \tag{5}$$

   What are the operators $\mathcal{C}\mathcal{O}(x)\mathcal{C}$ and $\mathcal{C}\mathcal{O}^\dagger(x)\mathcal{C}$?
7. Finally, we will also need reflection symmetry $\mathcal{P}$. It is defined by:

   $$\mathcal{P}^2 = 1, \qquad \mathcal{P}\Psi^\dagger_{R\sigma}(x)\mathcal{P} = \Psi^\dagger_{L\sigma}(-x), \qquad \mathcal{P}\Psi^\dagger_{L\sigma}(x)\mathcal{P} = \Psi^\dagger_{R\sigma}(-x). \tag{6}$$

   What are the operators $\mathcal{P}\mathcal{O}(x)\mathcal{P}$ and $\mathcal{P}\mathcal{O}^\dagger(x)\mathcal{P}$?

5 Symmetries and response functions

For a system with an unperturbed Hamiltonian H_0 and a time dependent perturbation $\delta H(t) = \lambda(t)B(t)$, the response function $R_{AB}(t-t')$ is defined by:

$$\delta \langle A \rangle (t) = \int dt' \, R_{AB}(t-t')\lambda(t'), \tag{7}$$

where $\delta \langle A \rangle (t)$ denotes the change in the expectation value of the observable A at time t induced by the perturbation. We recall that if the system is initially in its ground-state $|\Psi_0\rangle$, we have:

$$R_{AB}(t-t') = -i \langle \Psi_0 | [A(t), B(t')] | \Psi_0 \rangle \, \theta(t-t'), \tag{8}$$

where $A(t)$ and $B(t')$ are in the interaction picture with respect to H_0. We suppose that there is a symmetry operation, described by the unitary operator U, which leaves both H_0 and $|\Psi_0\rangle$ invariant, i.e. $UH_0 = H_0U$ and $U|\Psi_0\rangle = |\Psi_0\rangle$. We may also assume that A and B depend on a spatial coordinate, in which case we will write $A(x)$ and $B(x)$. Then, it is possible to expand these operators in Fourier modes:

$$A(x) = \sum_q e^{iqx} A_{-q}, \qquad B(x) = \sum_q e^{iqx} B_{-q} \tag{9}$$

Here A_q and B_q carry momentum q. This means that the action of A_q or B_q on any state with momentum k produces a state with momentum $k+q$.

8. If we denote $A' = UAU^{-1}$, and $B' = UBU^{-1}$, then show that $R_{AB}(t-t') = R_{A'B'}(t-t')$.
9. Write down a spectral decomposition of the Fourier transform (with respect to both space and time) of $R_{A(x)A(0)}$.
10. We suppose that the unitary symmetry operator U satisfies $UA_qU^{-1} = \tau A_{-q}$, with $\tau = \pm 1$. Show that $R_{AA}(q,\omega)$ is even in q, that $\Im R_{AA}(q,\omega)$ is odd in ω and that $\Re R_{AA}(q,\omega)$ is even in ω.

6 Mean-field approximation

Using renormalization group analysis in the second part of this problem (in some weeks), we will see that g_1 is irrelevant for positive and relevant for negative values, giving rise to a strong coupling phase. Anticipating this result, we will now attempt to understand better this strong coupling phase.

In this part, we set $g_2 = 0$ to simplify the discussion, and we write g for g_1. The mean-field approximation amounts, as usual, to replace the quartic operator $:\mathcal{O}\mathcal{O}^\dagger:$ by the combination $\left\langle \mathcal{O}^\dagger \right\rangle \mathcal{O} + \langle \mathcal{O} \rangle \mathcal{O}^\dagger$, supposing that the average is independant of x.

11. Why is the mean-field approximation also called large-N approximation ?
12. Introducing $\Delta = g\langle \mathcal{O} \rangle = |\Delta| e^{i\phi}$, determine the energy spectrum of the mean-field Hamiltonian H_{MF}. What are its properties ?

13. Check that $H_{\rm MF}$ is diagonalized in the quasiparticle basis whose associated creation operators are given by:

$$d^\dagger_{+\sigma}(k) = \sin\frac{\theta_k}{2}e^{-i\phi/2}c^\dagger_{R\sigma}(k) + \cos\frac{\theta_k}{2}e^{i\phi/2}c^\dagger_{L\sigma}(k) \tag{10}$$

$$d^\dagger_{-\sigma}(k) = \cos\frac{\theta_k}{2}e^{-i\phi/2}c^\dagger_{R\sigma}(k) - \sin\frac{\theta_k}{2}e^{i\phi/2}c^\dagger_{L\sigma}(k) \tag{11}$$

Here, we have introduced the angle θ_k defined by:

$$\cos\theta_k = -\frac{k}{\sqrt{k^2+|\Delta|^2}}, \quad \sin\theta_k = \frac{|\Delta|}{\sqrt{k^2+|\Delta|^2}} \tag{12}$$

14. Write explicitely the self-consistency equation for the parameter Δ.
15. Does the phase ϕ enter in the self-consistency equation ? What can we say from the symmetries of the mean-field ground state ? How is this related to the phase ϕ ?
16. Building from the previous question, by which physical argument are the mean-field solutions discredited ?
17. For which values of g is there a non-trivial solution, i.e. with $\Delta \neq 0$? Evaluate $|\Delta|$ in the thermodynamical limit as a function of g using two hypothesis:
    - we introduce an ultra-violet cut-off on allowed momenta:. we impose $|k| \leq \Lambda$,
    - we assume that the coupling is not too large, so that $|\Delta| \ll \Lambda$.

7 Collective modes

In this section, we shall consider the collective excitations of the system in the vicinity of the self-consistent mean-field ground-state with $\phi = 0$. This means that $\langle \mathcal{O} \rangle$ is real. In the sequel, it will be useful to distinguish between *amplitude* and *phase* fluctuations of the *order-parameter* $\langle \mathcal{O} \rangle$. For this, we define two hermitian components:

$$\mathcal{O}_a(x) = \frac{1}{2}(\mathcal{O}(x) + \mathcal{O}^\dagger(x)), \qquad \mathcal{O}_b(x) = \frac{i}{2}(\mathcal{O}^\dagger(x) - \mathcal{O}(x)) \tag{15}$$

The a-direction in order parameter plane is then associated to amplitude fluctuations and the b-direction to phase fluctuations.

18. Write the interaction term in $H_{\rm GN}$ with $\mathcal{O}_a(x)$ and $\mathcal{O}_b(x)$. What is now the mean-field Hamiltonian ?

    We will study collective modes in the spirit of the *Random Phase Approximation* (RPA). For this, we impose to the system, initially in a mean-field ground-state of $H_{\rm GN}$, an external perturbation:

$$\delta H(t) = \int_0^L dx\,(h_{{\rm ext},a}(x,t)\mathcal{O}_a(x) + h_{{\rm ext},b}(x,t)\mathcal{O}_b(x)) \tag{16}$$

    The perturbation will modify the expectations values of $\mathcal{O}_a$ and $\mathcal{O}_b$. As usual in the RPA, we take interactions into account by a space and time dependent deformation of the self-consistent field acting on the underlying particles. The RPA assumes then that the fermions respond as if they followed the mean-field Hamiltonian $H_{\rm MF}$, in the presence of local fields $h_{{\rm loc},a}, h_{{\rm loc},b}$, where:

$$h_{{\rm loc},i} = h_{{\rm ext},i} + 2Ng\delta\left\langle \mathcal{O}_i(x)\right\rangle . \tag{17}$$

In order to relate $\delta\left\langle \mathcal{O}_a(x)\right\rangle$ and $\delta\left\langle \mathcal{O}_b(x)\right\rangle$ to local fields, we use response functions $R_{ij}(q,\omega) \equiv R_{\mathcal{O}_i,\mathcal{O}_j}(q,\omega)$, $(i,j = a$ or $b)$, evaluated in the ground-state of the *spinless* version of $H_{\rm MF}$. This choice allows a simple tracking of the N-dependency of the order-parameter dynamics. In clear, we assume:

$$\delta\left\langle \mathcal{O}_i(x)\right\rangle(q,\omega) = \frac{1}{N}\sum_{j=a,b} R_{ij}(q,\omega)h_{{\rm loc},j}(q,\omega) \tag{18}$$

19. Explain the $1/N$ factor in the previous equation.

20. Give general expressions for the local fields $h_{{\rm loc},i}(q,\omega)$ and the order parameter fluctuations $\delta\left\langle \mathcal{O}_i(x)\right\rangle(q,\omega)$ in terms of the external fields $h_{{\rm ext},j}(q,\omega)$. In particular, the latter relations will serve to define dressed response functions $R^{\rm RPA}_{ij}(q,\omega)$ in the RPA.

21. What happens to $R^{\rm RPA}_{ij}(q,\omega)$ in the $N\to\infty$ limit ? Does this sound reasonable ?

22. What are the operators $\mathcal{C}\mathcal{O}_a(x)\mathcal{C}$ and $\mathcal{C}\mathcal{O}_b(x)\mathcal{C}$?

23. Show that H_{MF} for $\phi = 0$ commutes with the particle-hole symmetry operator $\mathcal{C}$ and that its ground-state is invariant under $\mathcal{C}$.

24. What can we infer from this for $R_{ab}(q,\omega)$?

25. What are the resulting dressed response functions $R^{\rm RPA}_{ij}(q,\omega)$?

26. Another very interesting phenomenon appears: there is a pole in $R^{\rm RPA}_{bb}(q,\omega)$ at $(q,\omega) = (0,0)$. Show that this is not a coincidence and that the existence of this pole can be predicted.

27. Show that H_{MF} for $\phi = 0$ commutes with the reflection symmetry operator $\mathcal{P}$ and that its ground-state is invariant under $\mathcal{P}$.

28. Show that $R_{aa}(q,\omega)$ and $R_{bb}(q,\omega)$ are even functions of q and even functions of ω.

29. Show that $\Im R_{aa}(q,\omega)$ and $\Im R_{bb}(q,\omega)$ are identically zero for ω not too large.

30. The previous remarks show that the Taylor expansion of $R_{aa}(q,\omega)$ and $R_{bb}(q,\omega)$ near $(q,\omega) = (0,0)$ has then the form $R_{ii}(q,\omega) = R_{ii}(0,0) - \lambda_i\omega^2 + \mu_i q^2 + ...$, $i = a,b$.

    The coefficients $R_{ii}(0,0), \lambda_i, \mu_i$ are real numbers. Show that $\lambda_i > 0$.

31. We will assume that $\mu_i > 0$. Show that it is the case if we neglect the variations of the matrix elements of $\mathcal{O}_{i,q}$ with q, $\langle 0|\mathcal{O}_{i,q}|\alpha_q\rangle$, where $|\alpha_q\rangle$ is a particle-hole excited state of momentum q.

32. A complete calculation shows that $R_{bb}(0,0) < R_{aa}(0,0) < 0$. Deduce from this informations and from the previous questions the poles of $R_{ii}(q,\omega)$ near $(\omega,q) = (0,0)$.

33. We define the symmetrized correlation function $C_{ii}(x,t)_s$ by:

$$C_{ii}(x,t)_s \equiv \frac{1}{2}\left(\langle \mathcal{O}_i(x,t)\mathcal{O}_i(0,0)\rangle + \langle \mathcal{O}_i(0,0)\mathcal{O}_i(x,t)\rangle\right). \tag{19}$$

The fluctuation-dissipation relation at zero temperature states that the Fourier transform of this correlation function is related to the corresponding response function by:

$$C_{ii}(q,\omega)_s = -\Im R_{ii}(q,\omega)\mathrm{sign}(\omega). \tag{20}$$

Use it to estimate the qualitative behavior of the correlation functions $C_{ii}(x,t=0)_s$. How do these results precise the physical picture for the low energy dynamics in the large N limit ?