Appendices

A The solution of the NLS equation

We consider the Non Linear Schrödinger equation

$$i\frac{\partial}{\partial\tau}F + P\frac{\partial^2}{\partial s^2}F + Q|F|^2F = 0 \tag{A.1}$$

and we look for solutions of the form

$$F(s,\tau) = a(s,\tau)\exp\big(i\theta(s,\tau)\big) \tag{A.2}$$

where a and θ are real functions. Introducing (A.2) into (A.1) and separating the real and immaginary parts we obtain

$$-a\frac{\partial\theta}{\partial\tau} + P\Big(\frac{\partial^2 a}{\partial s^2} - a\big(\frac{\partial\theta}{\partial s}\big)^2\Big) + Qa^3 = 0 \tag{A.3}$$

$$\frac{\partial a}{\partial\tau} + P\Big(2\frac{\partial a}{\partial s}\frac{\partial\theta}{\partial s} + a\frac{\partial^2\theta}{\partial s^2}\Big) = 0\,. \tag{A.4}$$

We look then for solutions characterized by the two relations

$$a = a(s - u_e\tau) \tag{A.5}$$

$$\theta = \theta(s - u_e\tau) \tag{A.6}$$

where u_e and u_c are respectively the envelope and carrier velocities. We obtain

$$\frac{\partial^2 a}{\partial s^2} - a\big(\frac{\partial\theta}{\partial s}\big)^2 + \frac{u_c}{P}a\frac{\partial\theta}{\partial s} + \frac{Q}{P}a^3 = 0 \tag{A.7}$$

$$a\frac{\partial^2\theta}{\partial s^2} + 2\frac{\partial a}{\partial s}\frac{\partial\theta}{\partial s} - \frac{u_e}{P}\frac{\partial a}{\partial s} = 0\,. \tag{A.8}$$

We introduce the notation ${u'}_e = \frac{u_e}{P}$, ${u'}_c = \frac{u_c}{P}$ and $Q' = \frac{Q}{P}$ in the two previous equations. Equation (A.8) becames then, after integration on s:

$$a^2\Big(2\frac{\partial\theta}{\partial s} - {u'}_e\Big) = C\,. \tag{A.9}$$

We are interested in localized solutions, then we must impose $\lim_{|s|\to\infty} a = 0$, and consequently $C = 0$: (A.9) gives then

$$\frac{\partial \theta}{\partial s} = \frac{u'_e}{2} \,. \tag{A.10}$$

Substituting $\frac{\partial \theta}{\partial s}$ into (A.7) we obtain

$$\frac{\partial^2 a}{\partial s^2} - a\frac{u'^2_e}{4} + a\frac{u'_e u'_c}{2} + Q' a^3 = 0 \;: \tag{A.11}$$

multiplying to $2\frac{\partial a}{\partial s}$ and integrating this gives

$$\Big(\frac{\partial a}{\partial s}\Big)^2 - a^2\frac{u'^2_e}{4} + a^2\frac{u'_e u'_c}{2} + Q'\frac{a^4}{2} = C' \,. \tag{A.12}$$

We have again to impose $C' = 0$, so that the previous equation can be written as

$$\Big(\frac{\partial a}{\partial s}\Big)^2 = V(a) \tag{A.13}$$

where

$$V(a) = \frac{a^2}{4}\Big(u'^2_e - 2u'_e u'_c\Big) - Q'\frac{a^4}{2} \,, \tag{A.14}$$

and, from (A.13),

$$\int_{a(0,0)}^{a(s,\tau)} \frac{da}{\sqrt{V(a)}} = s - u_e\tau \,. \tag{A.15}$$

We have then, putting $a_0 = \sqrt{\frac{u'^2_e - 2u'_e u'_c}{2Q'}}$, the solution

$$F(s,\tau) = a_0 \,\mathrm{sech}\left(\sqrt{\frac{Q'}{2}}\, a_0 \,(s - u_e\tau)\right) \exp\left(i\frac{u'_e}{2}\,(s - u_c\tau)\right) , \tag{A.16}$$

which can be rewritten in terms of the initial quantities introducing

$$A = \sqrt{\frac{u_e^2 - 2u_e u_c}{2PQ}} \tag{A.17}$$

$$L_e = \frac{2P}{\sqrt{u_e^2 - 2u_e u_c}} \tag{A.18}$$

giving finally

$$F(s,\tau) = A \,\mathrm{sech}\left(\sqrt{\frac{1}{L_e}}(s - u_e\tau)\right) \exp\left(i\frac{u_e}{2P}(s - u_c\tau)\right) . \tag{A.19}$$

B The nonlinearity matrices of second ($c^{\nu}_{d,k}$) and third order ($C^{\nu}_{d,k,j}$) for the *twist-opening* model

We remind that the nonlinear coefficient matrices (with two and three indices) for the system of equations of motion (2.39), (2.40), $c^{\nu}_{d,k}(\nu',\nu'')$, and $C^{\nu}_{d,k,j}(\nu',\nu'',\nu''')$, represent the coefficients in such a system of the quadratic and cubic terms in the field $\vec{E}(n,t) = \big(y_n(t), \phi_n(t)\big)$ and its derivatives. Index ν corresponds to the number of the equation, ν', ν'' and ν''' to the field components involved in the quadratic and cubic term considered, and represents the matrix indexes. d is the overall derivative number of the term and k, j are introduced to keep into account the different terms that contains the same total number of derivatives.

The second order nonlinearities in the equations of motion for the *twist-opening* model correspond to the following matrices of coefficients:

$$
\begin{aligned}
& c^1_{0,0} = \begin{pmatrix} 3/2 & 0 \\ 0 & 0 \end{pmatrix} && c^2_{0,0} = 0 \\
& c^1_{1,i} = 0 \qquad \text{for } i = 0,1 && c^2_{1,i} = 0 \qquad \text{for } i = 0,1 \\
& c^1_{2,1} = \begin{pmatrix} 0 & 0 \\ 0 & 1/R_0 \end{pmatrix} && c^2_{2,0} = c^2_{2,1} = \begin{pmatrix} 0 & -2/R_0 \\ 0 & 0 \end{pmatrix} \\
& c^1_{2,i} = 0 \qquad \text{for } i = 0,2 && c^2_{2,2} = 0\,.
\end{aligned}
\tag{B.1}
$$

The nonzero components in the previous matrices all derive from kinetic nonlinearities in the equations of motion (2.39), (2.40), except for the coefficient 3/2 in $c^1_{0,0}$ which corresponds to the second order Morse nonlinearity in y_n.

The third order nonlinearities $C^{\nu}_{d,k,j}(\nu',\nu'',\nu''')$ in equations (2.39), (2.40) are matrices of coefficients with three indices. We just list the nonzero components:

$$
\begin{aligned}
& C^1_{0,0,0}(1,1,1) = -7/6 && C^1_{2,1,0}(1,2,2) = 1/R^2 \\
& C^2_{2,0,0}(1,1,2) = -1/R^2 && C^2_{2,1,0}(1,1,2) = -2/R^2\,.
\end{aligned}
\tag{B.2}
$$

The first nonzero coefficient arise fron the Morse nonlinear terms of type $-7/6\ y^3$, the others, respectively, from the kinetic terms $1/R^2\ y_n\dot{\phi}_n^{\,2}$ in the first equation, $-1/R^2\ y_n^2\ddot{\phi}_n$ and $-2/R^2\ y_n\dot{y}_n\dot{\phi}_n$ in the second equation.

C Coefficients for the correction terms and value of Q

We report here the coefficients $\vec{\gamma}_c$, μ_{1c} , σ_{2c} , μ_{2c} used in Section 3.3 to define the second harmonic and d.c. corrections in (3.38). Their calculation is straightforward as explained below. It is useful to define the quantities

$$\Delta = \left|\left|\hat{J}(2q) - 4\omega^2\right|\right| = (a(2q) - 4\omega_+^2)(b(2q) - 4\omega_+^2) - |c(2q)|^2 \tag{C.1}$$

and

$$\mathcal{H} = \frac{3}{2}|V_1^+|^2 - \frac{\omega_+^2 |V_2^+|^2}{R_0} \tag{C.2}$$

$$\mathcal{K} = \frac{4i\omega_+^2 V_1^+ V_2^+}{R_0} \,. \tag{C.3}$$

Coefficents giving $\vec{\gamma}_c$ are then:

$$\gamma_{1c} = \frac{\mathcal{H}(b(2q) - 4\omega_+^2) - \mathcal{K}c(2q)}{\Delta} \tag{C.4}$$

$$\gamma_{2c} = \frac{\mathcal{K}(a(2q) - 4\omega_+^2) + \mathcal{H}c^*(2q)}{\Delta} \tag{C.5}$$

More explicit values could be easly write down through the substitutions

$$\begin{aligned} a(2q) &= 1 + 4K_{yy}\cos{(2q)}^2 \\ b(2q) &= 4K_{\phi\phi}\sin{(2q)}^2 + G\big(6 - 8cos(2q) + 2\cos{(4q)}\big) \\ c(2q) &= K_{y\phi}\,\sin{(2q)}\,. \end{aligned} \tag{C.6}$$

Coefficients for the non oscillating terms μ_{1c} and σ_{2c} derive from system (3.47). Introducing the quantities

$$\mathcal{J} = 3|V_1^+|^2 + \frac{2\omega_+^2}{R_0}|V_2^+|^2 \tag{C.7}$$

$$\mathcal{L} = \frac{-2i\omega_+\omega_+^{(1)}}{R_0}\Big(V_2^* V_1 - V_2 V_1^*\Big) \tag{C.8}$$

and using the result

$$\left|\left|\begin{pmatrix} a(0) & -ic'(0) \\ -ic'^*(0) & -b''(0)/2 \end{pmatrix}\right|\right| = -\frac{b''(0)}{2}\,, \tag{C.9}$$

where $b''(0) = -b(0) = 2K_{\phi\phi}$, we get

$$\mu_{1c} = \mathcal{J} + \frac{2ic'(0)}{b(0)}\mathcal{L} = \mathcal{J} + \frac{K_{y\phi}}{K_{\phi\phi}}\mathcal{L} \tag{C.10}$$

$$\sigma_{2c} = \frac{2a(0)}{b(0)}\mathcal{L} + \frac{2ic'^*(0)}{b(0)}\mathcal{J} = -\frac{1+4K_{yy}}{K_{\phi\phi}}\mathcal{L} - \frac{K_{y\phi}}{K_{\phi\phi}}\mathcal{J} \tag{C.11}$$

The value of μ_{2c} in equation (3.50) is calculated directly by integration of (3.45), using the amplitude function (3.54). It depends on parameters involved in the definition of the amplitude (namely P, u_e):

$$\mu_{2c} = \frac{2\omega\omega^{(1)}}{K_{y\phi}PR_0}|V_2^+|^2 u_e \,. \tag{C.12}$$

The value of $\vec{\mathcal{Q}}/2\omega_+ = (Q_1, Q_2)$ in (3.53) is given by:

$$\begin{aligned} Q_1 &= -\frac{7}{2}|V_1^+|^2V_1^+ + \frac{\omega_+^2}{R_0^2}(2V_1^+|V_2^+|^2 - V_1^{+*}V_2^{+2}) + \\ &\quad 3V_1^{+*}\gamma_{1c} + \frac{4\omega_+^2}{R_0}V_2^{+*}\gamma_{2c} + 3V_1^+\mu_{1c} & \text{(C.13)} \\ Q_2 &= \frac{\omega_+^2}{R_0}(4V_1^{+*}\gamma_{2c} - 2V_2^{+*}\gamma_{1c}) + \frac{2\omega_+^2}{R_0}V_2^+\mu_{1c} - \\ &\quad \frac{2\omega_+^2}{R_0^2}V_1^{+2}V_2^{+*} + \frac{\omega_+^2}{R_0^2}(2|V_1^+|^2V_2^+ + V_1^{+2}V_2^{+*}) & \text{(C.14)} \end{aligned}$$