# Symmetries of non-linear ODEs: lambda extensions of the Ising correlations 

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#### Abstract

. This paper provides several illustrations of the numerous remarkable properties of the lambda-extensions of the two-point correlation functions of the Ising model, sheding some light on the non-linear ODEs of the Painlevé type. We first show that this concept also exists for the factors of the two-point correlation functions focusing, for pedagogical reasons, on two examples namely $C(0,5)$ and $C(2,5)$ at $\nu=-k$. We then display, in a learn-by-example approach, some of the puzzling properties and structures of these lambda-extensions: for an infinite set of (algebraic) values of $\lambda$ these power series become algebraic functions, and for a finite set of (rational) values of lambda they become D-finite functions, more precisely polynomials (of different degrees) in the complete elliptic integrals of the first and second kind $K$ and $E$. For generic values of $\lambda$ these power series are not D-finite, they are differentially algebraic. For an infinite number of other (rational) values of $\lambda$ these power series are globally bounded series, thus providing an example of an infinite number of globally bounded differentially algebraic series. Finally, taking the example of a product of two diagonal two-point correlation functions, we suggest that many more families of non-linear ODEs of the Painlevé type remain to be discovered on the two-dimensional Ising model, as well as their structures, and in particular their associated lambda extensions. The question of their possible reduction, after complicated transformations, to Okamoto sigma forms of Painlevé VI remains an extremely difficult challenge.


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## 1. Introduction: linear versus non-linear symmetry representations

It is not necessary to underline the fundamental role played by the concept of symmetry in physics [1], or in applied mathematics, and in the foundations for the fundamental theories of modern physics. Symmetries can correspond to continuous or
discrete transformations, and are frequently amenable to mathematical formulations such as group representations, with invariant or covariant properties, non-trivial identities, conservation laws, ...

Integrable models (in dynamical systems, lattice statistical mechanics, quantum field theory, solid state physics, enumerative combinatorics, ...) play a selected role, since they correspond to situations where one has "enough" (possibly an infinite number of) conserved quantities to solve the problem. We are not going to recall the techniques and tools introduced to achieve that goal (Yang-Baxter equations, Bethe Ansatz, Lax pairs, Schlesinger systems [2] ,..) but we will rather focus on the linear and non-linear differential equations emerging naturally in these problem, and on the corresponding symmetries of these ordinary differential equations. To address that problem we will, for pedagogical reasons, focus on the analysis of the two-point correlation functions of a fundamental integrable model, the two-dimensional Ising model [6].

Some two-point correlation functions $C(M, N)$ of the two-dimensional Ising model can be seen as solutions of linear differential equations and, in the same time, also as solutions of non-linear differential equations, namely Okamoto sigma-forms of Painlevé VI equations. The solutions of these last non-linear ODEs naturally introduce one-parameter families of power series solutions, that are called lambda-extensions of the two-point correlation functions.

The two-point correlation functions $C(M, N)$ we will consider [7, 8] for the special case $\nu=-k$ (or in the isotropic case $\nu=1$ ), are polynomial expressions of the complete elliptic integrals of the first and second kind $K$ and $E$ : they are solutions of linear differential operators with polynomial coefficients, in other words they are $D$-finite, however, when introducing some well-suited log-derivative of these two-point correlation functions (see (4) below), they are also solutions of highly selected non-linear differential equations having the Painlevé property [9, namely Okamoto sigma-forms 10 of Painlevé VI (see (5) below), in other words they are differentially algebraid $\ddagger$. The two-point correlation functions $C(M, N)$ have in the same time, a linear (D-finite) description and a non-linear (differentially algebraic) description! The question of the analysis of the symmetries of these two linear and non-linear ordinary differential equations, and of the symmetries of their solutionst十 naturally pops out. It is crucial to note that the non-linear ordinary differential equations for the two-point correlation functions $C(M, N)$ correspond to one closed equation (see (5) below) where the two integers $M$ and $N$ are parameters in the equation. In contrast the linear differential equations for the $C(M, N)$ correspond to an infinite number of linear differential equations of order (and degree and size) growing with the two integers $M$ and $N$. Each description (linear versus non-linear) has its own advantages and disadvantages: an infinite number of differential operators to be discovered but they are simply linear, versus one ( $M, N$-dependent) equation

[^0]encapsulating everything, but it is non-linear. The analysis of the symmetries of the linear differential operators associated with the two-point correlation functions $C(M, N)$ can, for instance, be performed considering the corresponding differential Galois group. Actually we have seen in previous papers 13 that the linear differential operators emerging in the integrable models are systematically associated with selected differential Galois groups, the operators being homomorphic [13] to their adjoint associated operatorsti. In this Ising case, the linear differential operators are homoomorphic [14] to the symmetric $N$-th power of the order-two linear differential operator annihilating the complete elliptic integrals of the first or second kind $K$ and $E$. Along this line, some mathematicians could argue that, if a differential Galois group approach of integrability is probably natural, an extension of the concept of differential Galois group for non-linear ODEs is certainly hopeless in general [15. They may even argue (see [15] in section 6.2) that, even if most of the people that work in integrability consider the families of Painlevé transcendents [16, 17 as integrable, their opinion is that, in general, they are non integrable. Let us recall that the sigma-form of Painlevé VI equations (like (5) below), are highly selected non-linear ODEs: they have the fixed critical point property [20, 21, 22] (Painlevé property) and can be obtained from isomonodromic deformations of linear differential equations [23, 24, which allows to see these non-linear ODEs as compatibility conditions of a linear Schlesinger system of PDEs. In that case one could imagine to consider a differential Galois Theory for the underlying Schlesinger system. The purpose of this paper is not to build a differential Galois Theory of Painlevé equations in order to discuss, from a very general mathematical viewpoint the "symmetries" of the non-linear ODEs (like (5) below) emerging for the $C(M, N)$ Ising two-point correlation functions. On the contrary, in a very pedagogical, learn-by-examples approach, we will display a large set of the properties (symmetries ...) of the $C(M, N)$ two-point correlation functions, with a focus on the remarkable propertiestly of the lambda-extensions solutions of the sigmaform of Painlevé VI non-linear ODEs (like (5) below). For pedagogical reasons we will restrict to $C(0,5)$ and $C(2,5)$. Then, taking an example of product of two diagonal two-point correlation functions, we will suggest that many more families of non-linear ODEs of the Painlevé type remain to be discovered on the two-dimensional Ising model, as well as their structures, and in particular their associated lambda extensions. Finally, we will give additional comments and results providing an illustration of a set of remarkable, and sometimes puzzling, properties of the lambda-extensions of the Ising two-point correlation functions.

## 2. Recalls

We revisit, with a pedagogical heuristic motivation, the lambda extensions [14 of some two-point correlation functions $C(M, N)$ of the two-dimensional Ising model. For simplicity we will examine in detail the lambda extensions of a particular lowtemperature diagonal correlation function, namely $C(0,5)$ and $C(2,5)$, in order to make crystal clear some structures and subtleties. Note however that similar structures
$\dagger$ We even have this remarkable property with most of the linear differential operators annihilating diagonals of rational functions 13 .
§ At least in the (narrow) Liouville sense [18, 19.
$\dagger \dagger$ We must also mention the fact that the lambda-extensions of the two-point correlation functions $C(M, N)$ also verify quadratic discrete recursions 25, 26, 27] (lattice recursions in the two integers $M$ and $N$ ), that can be seen as integrable lattice recursions
and results can also be obtained on other two-point correlation functions $C(M, N)$ for the special case $\nu=-k$ studied in [7] where Okamoto sigma-forms of Painlevé VI equations also emerge.

In a previous paper [7] we considered the two-point correlation $C(M, N)$ of spins at sites $(0,0)$ and $(M, N)$, of the anisotropic Ising model defined by the interaction energy

$$
\begin{equation*}
\mathcal{E}=-\sum_{j, k}\left\{E_{v} \sigma_{j, k} \sigma_{j+1, k}+E_{h} \sigma_{j, k} \sigma_{j, k+1}\right\} \tag{1}
\end{equation*}
$$

where $\sigma_{j, k}= \pm 1$ is the spin at row $j$ and column $k$, and where the sum is over all lattice sites. Defining

$$
\begin{equation*}
k=\left(\sinh 2 E_{v} / k_{B} T \sinh 2 E_{h} / k_{B} T\right)^{-1} \quad \text { and } \quad \nu=\frac{\sinh 2 E_{h} / k_{B} T}{\sinh 2 E_{v} / k_{B} T} \tag{2}
\end{equation*}
$$

we found [7] that in the special case

$$
\begin{equation*}
\nu=-k \tag{3}
\end{equation*}
$$

the correlation $C(M, N)$ satisfies an Okamoto sigma-form of the Painleve VI equation.

$$
\text { For } T<T_{c}, M \leq N \text { and } \nu=-k, \text { with } t=k^{2}, \text { introducing }
$$

$$
\begin{equation*}
\sigma=t \cdot(t-1) \cdot \frac{d \ln C(M, N)}{d t}-\frac{t}{4} \tag{4}
\end{equation*}
$$

we have, when $M+N$ is odd, the following Okamoto sigma-form of the Painlevé VI equation [7]:

$$
\begin{align*}
& t^{2} \cdot(t-1)^{2} \cdot \sigma^{\prime \prime 2}+4 \cdot \sigma^{\prime} \cdot\left(t \sigma^{\prime}-\sigma\right) \cdot\left((t-1) \cdot \sigma^{\prime}-\sigma\right) \\
& \quad-M^{2} \cdot\left(t \sigma^{\prime}-\sigma\right)^{2}-N^{2} \cdot \sigma^{\prime 2}+\left(M^{2}+N^{2}\right) \cdot \sigma^{\prime} \cdot\left(t \sigma^{\prime}-\sigma\right)=0 \tag{5}
\end{align*}
$$

### 2.1. Two factors

In this $M+N$ odd, $M \leq N, M \neq 0, \nu=-k$ case, the correlation functions factor into two factors. We will write the factorizations of these $C(M, N)$ 's as

$$
\begin{equation*}
(1-t)^{-1 / 4} \cdot C(M, N ; t)=g_{+}(M, N ; t) \cdot g_{-}(M, N ; t) \tag{6}
\end{equation*}
$$

where the two factors $g_{ \pm}$are homogeneous polynomials of the complete elliptic integrals of the first and second kind:

$$
\begin{align*}
\tilde{K}(k) & =\frac{2}{\pi} \cdot K(k)={ }_{2} F_{1}\left(\left[\frac{1}{2}, \frac{1}{2}\right],[1], k^{2}\right) \\
\tilde{E}(k) & \left.=\frac{2}{\pi} \cdot E(k)={ }_{2} F_{1}\left(\frac{1}{2},-\frac{1}{2}\right],[1], k^{2}\right) \tag{7}
\end{align*}
$$

We consider the following logarithmic derivatives of the previous two factors:

$$
\begin{equation*}
\sigma_{ \pm}(M, N ; t)=t \cdot(t-1) \cdot \frac{d \ln g_{ \pm}(M, N ; t)}{d t} \tag{8}
\end{equation*}
$$

ब The condition $\nu=-k$ (as well as the isotropic case $\nu=1$ ) is special because it is such that the complete elliptic integrals of the third kind (EllipticPi in Maple), appearing in the anisotropic case, reduce to complete elliptic integrals of the second kind (see equation (30) in 7]).
$\dagger$ Which is the same as the Toeplitz determinants [28] of Forrester-Witte [29] as given in 30].
$\dagger \dagger$ What is written here in subsection 2.1 is also true when $M=0$, with the caveat that $g_{+}$and $g_{-}$ in (6) factor into two factors (see subsection 2.2 below).

The sigma functions have additive decompositions which follow from the mulplicative decompositions (6):

$$
\begin{equation*}
\sigma(M, N ; t)=\sigma_{+}(M, N ; t)+\sigma_{-}(M, N ; t) \tag{9}
\end{equation*}
$$

Here we begin with the factorizations (6) of the $C(M, N)$ 's with $M+N$ odd, $M \leq N$, for miscellaneous values of $M$ and $N$, and, by use of the methods described in [7] and of the program guessfunc of Jay Pantone [31], we find that both $\sigma_{+}(M, N ; t)$ and $\sigma_{-}(M, N ; t)$ in (9) satisfy the same second-order non-linear differential equation

$$
\begin{align*}
& 32 t^{3} \cdot(t-1)^{2} \cdot \sigma^{\prime \prime 2}+4 t^{2} \cdot(t-1) \cdot\left(8 \cdot \sigma-8 \cdot(t+1) \cdot \sigma^{\prime}+M^{2}-N^{2}\right) \cdot \sigma^{\prime \prime} \\
& -\left(8 \sigma-16 \cdot t \sigma^{\prime}+M^{2} t-N^{2}+1-t\right) \cdot\left(8 \cdot t \cdot(t-1) \cdot \sigma^{\prime 2}-16 t \cdot \sigma \cdot \sigma^{\prime}\right. \\
& \left.\quad+8 \cdot \sigma^{2}+\left(M^{2}-N^{2}\right) \cdot \sigma\right)=0 \tag{10}
\end{align*}
$$

where the prime indicates a derivative with respect to $t$, and where $\sigma$ is one of the two log-derivatives (8).

The two solutions (8) of (10), $\sigma_{+}(M, N ; t)$ and $\sigma_{-}(M, N ; t)$, have different boundary conditions. Note that $\sigma_{ \pm}=0$ is a selected solution of (10).

### 2.2. Four factors

In [7], we discovered that $C(0, N)$ with $N$ odd and $k=-\nu$, in the low-temperature regime, factors into four terms instead of two. The four factors for $C(0, N)$ were presented as

$$
\begin{equation*}
C(0, N)=\text { constant } \cdot(1-t)^{1 / 2} \cdot t^{\left(1-N^{2}\right) / 4} \cdot f_{1} f_{2} f_{3} f_{4} \tag{11}
\end{equation*}
$$

where the factors $f_{j}$ all vanish at $t=0$ in such a way to cancel the factor $t^{\left(1-N^{2}\right) / 4}$. We normalize the factors $f_{i}$ in (11) in such a way to extract a factor of $(1-t)^{1 / 4}$ which is the limiting behavior of $C(0, N)$ as $N \rightarrow \infty$, and we impose the condition that the four new factors satisfy the same non-linear differential equation. The previous factorization (11) in four factors now reads [8:

$$
\begin{equation*}
(1-t)^{-1 / 4} \cdot C(0, N)=g_{1}(0, N) \cdot g_{2}(0, N) \cdot g_{3}(0, N) \cdot g_{4}(0, N) \tag{12}
\end{equation*}
$$

If one defines

$$
\begin{equation*}
\sigma_{j}=t \cdot(t-1) \cdot \frac{d \ln g_{j}(t)}{d t} \tag{13}
\end{equation*}
$$

the previous factorization (12) in four factors becomes an additivity property of the corresponding $\sigma_{i}$ 's:

$$
\begin{equation*}
\sigma(0, N)=\sigma_{1}(0, N)+\sigma_{2}(0, N)+\sigma_{3}(0, N)+\sigma_{4}(0, N) \tag{14}
\end{equation*}
$$

These $\sigma_{i}$ 's are solutions of the same non-linear differential equation of the Painlevé type which reads:

$$
\begin{gather*}
t^{2} \cdot(t-1)^{2} \cdot \sigma^{\prime \prime 2}+4 \sigma^{\prime} \cdot\left(t \cdot \sigma^{\prime}-\sigma\right) \cdot\left((t-1) \cdot \sigma^{\prime}-\sigma\right) \\
+\frac{1}{4} \cdot\left(\left(N^{2}+1\right) \cdot(t-1)-t^{2}\right) \cdot \sigma^{\prime 2}-\frac{1}{2^{6}} \cdot\left(16 \cdot\left(N^{2}+1-2 t\right) \cdot \sigma+N^{2} \cdot t\right) \cdot \sigma^{\prime} \\
-\frac{1}{4} \cdot \sigma^{2}+\frac{N^{2}}{2^{6}} \cdot \sigma-\frac{N^{2} \cdot\left(N^{2}-3\right)}{2^{10}}=0 \tag{15}
\end{gather*}
$$

- Note that this second order non-linear ODE, which is actually of the Painlevé type, is not of the Okamoto sigma-form of Painlevé VI form, but it can be reduced to such a form using non trivial transformations (equations (26), (28) in section (2) of 8]).
$\dagger$ Examples of $g_{i}(0, N)$ 's for $C(0,5)$ and $C(0,7)$ are given in [8].

3. $\alpha$-extension of the four factors $f_{1}, f_{2}, f_{3}, f_{4}$ for $C(0,5)$

We underlined that the (low-temperature) row correlation functions $C(0, N)$ factor, when is $N$ odd, into four factors (11). These four factors $f_{i}$ 's are each a homogeneous polynomial of the complete elliptic functions $E$ and $K$. Furthermore one can see that each of these four factors is a Toeplitz determinant (see for instance section G. 4 of appendix $G$ in [8).

More specifically let us revisit the $N=5$ case detailed in [7] and also [8], where the two-point correlation $C(0,5)$ factors as follows

$$
\begin{equation*}
C(0,5)=\frac{256}{81} \cdot \frac{(1-t)^{1 / 2}}{t^{6}} \cdot f_{1} \cdot f_{2} \cdot f_{3} \cdot f_{4} \tag{16}
\end{equation*}
$$

where:

$$
\begin{align*}
f_{1}= & (2 t-1) \cdot \tilde{E}+(1-t) \cdot \tilde{K}, \quad f_{2}=(1+t) \cdot \tilde{E}-(1-t) \cdot \tilde{K}  \tag{17}\\
& f_{3}=(t-2) \cdot \tilde{E}+2 \cdot(1-t) \cdot \tilde{K}  \tag{18}\\
& f_{4}=3 \tilde{E}^{2}+2 \cdot(t-2) \cdot \tilde{E} \tilde{K}+(1-t) \cdot \tilde{K}^{2} \tag{19}
\end{align*}
$$

These exact polynomial expressions in terms of complete elliptic integrals of the first and second kind $\tilde{K}$ and $\tilde{E}$, actually have some lambda-extensions. Considering the non-linear ODE's verified by these $f_{n}$ 's one can, by a down-to-earth, order by order expansion of the analytic at $t=0$ solution, find the series expansion of a one parameter family of solution of the non-linear ODE's (we will denote $\alpha$ this parameter), such that $\alpha=0$ corresponds to the previous exact expressions (17), (18), (19). The first terms of these $\alpha$-dependent solutions read:

$$
\begin{align*}
f_{1}(\alpha)= & \frac{3}{2} t-\frac{9 t^{2}}{16}-\frac{15 t^{3}}{128}-\left(\frac{105}{2048}+\frac{15}{1024} \alpha\right) \cdot t^{4}-\left(\frac{945}{32768}+\frac{135}{8192} \alpha\right) \cdot t^{5} \\
- & \left(\frac{4851}{262144}+\frac{513}{32768} \alpha\right) \cdot t^{6}-\left(\frac{27027}{2097152}+\frac{7497}{524288} \alpha\right) \cdot t^{7}  \tag{20}\\
& -\left(\frac{637065}{67108864}+\frac{434295}{33554432} \alpha\right) \cdot t^{8} \\
& -\left(\frac{15643485}{2147483648}+\frac{6292455}{536870912} \alpha-\frac{105}{536870912} \alpha^{2}\right) \cdot t^{9}+\cdots \\
f_{2}(\alpha)= & \frac{3}{2} t-\frac{3 t^{2}}{16}-\frac{3 t^{3}}{128}-\left(\frac{15}{2048}-\frac{15}{1024} \alpha\right) \cdot t^{4}-\left(\frac{105}{32768}-\frac{165}{8192} \alpha\right) \cdot t^{5} \\
- & \left(\frac{441}{262144}-\frac{723}{32768} \alpha\right) \cdot t^{6}-\left(\frac{2079}{2097152}-\frac{11799}{524288} \alpha\right) \cdot t^{7}  \tag{21}\\
- & \left(\frac{42471}{67108864}-\frac{747927}{33554432} \alpha\right) \cdot t^{8} \\
& -\left(\frac{920205}{214748348}-\frac{11692785}{536870912} \alpha-\frac{105}{536870912} \alpha^{2}\right) \cdot t^{9}+\cdots \\
f_{3}(\alpha)= & -\frac{3}{8} t^{2}-\frac{3 t^{3}}{32}-\frac{45 t^{4}}{1024}-\frac{105 t^{5}}{4096}-\left(\frac{2205}{131072}-\frac{15}{131072} \alpha\right) \cdot t^{6} \\
& -\left(\frac{6237}{524288}-\frac{135}{524288} \alpha\right) \cdot t^{7}-\left(\frac{297297}{33554432}-\frac{3285}{8388608} \alpha\right) \cdot t^{8} \\
& -\left(\frac{920205}{134217728}-\frac{16965}{33554432} \alpha\right) \cdot t^{9}+\cdots \tag{22}
\end{align*}
$$

$$
\begin{align*}
f_{4}(\alpha)= & -\frac{3}{8} t^{2}-\frac{3}{16} t^{3}-\frac{129 t^{4}}{1024}-\frac{195 t^{5}}{2048}-\left(\frac{5025}{65536}+\frac{15}{131072} \alpha\right) \cdot t^{6} \\
- & \left(\frac{8421}{131072}+\frac{75}{262144} \alpha\right) \cdot t^{7}-\left(\frac{1856253}{33554432}+\frac{3975}{8388608} \alpha\right) \cdot t^{8} \\
& -\left(\frac{3260907}{67108864}+\frac{11025}{16777216}\right) \cdot t^{9}+\cdots \tag{23}
\end{align*}
$$

Furthermore one sees, on the series expansions of the $\alpha$-extensions (20), (21), (22), (23), the following remarkable identities

$$
\begin{array}{ll}
(1-t)^{1 / 4} \cdot f_{2}(\alpha)=f_{1}(1-\alpha), & (1-t)^{1 / 4} \cdot f_{2}(1-\alpha)=f_{1}(\alpha), \\
(1-t)^{1 / 4} \cdot f_{4}(\alpha)=f_{3}(1-\alpha), & (1-t)^{1 / 4} \cdot f_{4}(1-\alpha)=f_{3}(\alpha), \tag{24}
\end{array}
$$

and thus:

$$
\begin{align*}
& (1-t)^{1 / 2} \cdot f_{2}(\alpha) \cdot f_{4}(\alpha)=f_{1}(1-\alpha) \cdot f_{3}(1-\alpha), \\
& \left.(1-t)^{1 / 2} \cdot f_{2}(1-\alpha) \cdot f_{4}(1-\alpha)=f_{1}(\alpha) \cdot f_{3} \alpha\right),  \tag{25}\\
& f_{4}(\alpha) \cdot f_{1}(1-\alpha)=f_{2}(\alpha) \cdot f_{3}(1-\alpha), \\
& f_{4}(1-\alpha) \cdot f_{1}(\alpha)=f_{2}(1-\alpha) \cdot f_{3}(\alpha) . \tag{26}
\end{align*}
$$

In particular one has:

$$
\begin{align*}
& f_{1}(0)=(2 t-1) \cdot \tilde{E}+(1-t) \cdot \tilde{K}  \tag{27}\\
& f_{1}(1)=(1-t)^{1 / 4} \cdot((1+t) \cdot \tilde{E}-(1-t) \cdot \tilde{K})  \tag{28}\\
& f_{2}(0)=(1+t) \cdot \tilde{E}-(1-t) \cdot \tilde{K}  \tag{29}\\
& f_{2}(1)=(1-t)^{-1 / 4} \cdot((2 t-1) \cdot \tilde{E}+(1-t) \cdot \tilde{K})  \tag{30}\\
& f_{3}(0)=(t-2) \cdot \tilde{E}+2 \cdot(1-t) \cdot \tilde{K}  \tag{31}\\
& f_{3}(1)=(1-t)^{-1 / 4} \cdot\left(3 \tilde{E}^{2}+2 \cdot(t-2) \cdot \tilde{E} \tilde{K}+(1-t) \cdot \tilde{K}^{2}\right)  \tag{32}\\
& f_{4}(0)=3 \tilde{E}^{2}+2 \cdot(t-2) \cdot \tilde{E} \tilde{K}+(1-t) \cdot \tilde{K}^{2}  \tag{33}\\
& f_{4}(1)=(1-t)^{1 / 4} \cdot((t-2) \cdot \tilde{E}+2 \cdot(1-t) \cdot \tilde{K}) \tag{34}
\end{align*}
$$

It is worth noticing that (in contrast with the $\lambda$-extension $C(0,5 ; \lambda)$ see (35) below), the $f_{n}(\alpha)$ 's have two different values of the parameter $\alpha$ for which these $\alpha$-extensions are D-finite, being (homogeneous) polynomials in $\tilde{E}$ and $\tilde{K}$. One remarks with (31) and (32) (or (33) and (34)), that the corresponding polynomials in $\tilde{E}$ and $\tilde{K}$ are not necessarily of the same degree in $\tilde{E}$ and $\tilde{K}$.

The $\lambda$-extension $C(0,5 ; \lambda)$ solution of the same non-linear ODE verified by $C(0,5)$ (namely (5) for $N=5$ ) corresponds to the form-factor expansion [14, 32] which amounts to seeing this one-parameter family of solutions as a deformation of the $(1-t)^{1 / 4}$ algebraic solution of the previous non-linear ODE (5) verified by $C(0,5)$ :

$$
\begin{align*}
& C(0,5 ; \lambda)=(1-t)^{1 / 4} \cdot\left(1+\lambda^{2 n} \cdot \sum_{n=1}^{\infty} f_{0,5}^{2 n}\right)  \tag{35}\\
& \quad=1-\frac{t}{4}-\frac{3 t^{2}}{32}-\frac{7 t^{3}}{128}-\frac{77 t^{4}}{2048}-\frac{231 t^{5}}{8192}-\left(\frac{1463}{65536}+\frac{25}{1048576} \cdot \lambda^{2}\right) \cdot t^{6} \\
& \quad-\left(\frac{4807}{262144}+\frac{275}{4194304} \cdot \lambda^{2}\right) \cdot t^{7}-\left(\frac{129789}{8388608}+\frac{123475}{1073741824} \cdot \lambda^{2}\right) \cdot t^{8}+\cdots
\end{align*}
$$

### 3.1. Deformation around a D-finite solution.

The $\lambda$-extension of the correlation function $C(0,5 ; \lambda)$ can also be seen as a $\mu$ deformation of the series of the correlation $C(0,5)$ which exact expression is given by (16) (with (17), (18), (19)) in terms of polynomials in $\tilde{E}$ and $\tilde{K}$. This one-parameter $\mu$-family of series expansion which verifies the same non-linear ODE (5) as $C(0,5)$, reads:

$$
\begin{align*}
C(0,5 ; \lambda)= & C(0,5)+\mu \cdot G_{1}(t)+\mu^{2} \cdot G_{2}(t)+\mu^{3} \cdot G_{3}(t)+\cdots \\
=1- & -\frac{t}{4}-\frac{3 t^{2}}{32}-\frac{7 t^{3}}{128}-\frac{77 t^{4}}{2048}-\frac{231 t^{5}}{8192}-\left(\frac{23433}{1048576}-\frac{25}{1048576} \mu\right) \cdot t^{6} \\
- & \left(\frac{77187}{4194304}-\frac{275}{4194304} \mu\right) \cdot t^{7}-\left(\frac{16736467}{1073741824}-\frac{123475}{1073741824} \mu\right) \cdot t^{8} \\
& -\left(\frac{57930653}{4294967296}-\frac{708125}{4294967296} \mu\right) \cdot t^{9}+\cdots \tag{36}
\end{align*}
$$

The identification of these two power series (35) and (36) corresponds to the simple relation between $\lambda$ and $\mu$ :

$$
\begin{equation*}
\lambda^{2}=1-\mu \quad \text { or: } \quad \mu=1-\lambda^{2} \tag{37}
\end{equation*}
$$

This one-parameter series (35), or (36), is in agreement with a $\alpha$-extension of the four products formula (16)

$$
\begin{equation*}
C(0,5 ; \lambda)=\frac{256}{81} \cdot \frac{(1-t)^{1 / 2}}{t^{6}} \cdot f_{1}(\alpha) \cdot f_{2}(\alpha) \cdot f_{3}(\alpha) \cdot f_{4}(\alpha) \tag{38}
\end{equation*}
$$

if

$$
\begin{equation*}
\mu=4 \cdot \alpha \cdot(1-\alpha) \quad \text { or: } \quad \lambda^{2}=(2 \alpha-1)^{2} \tag{39}
\end{equation*}
$$

or:

$$
\begin{equation*}
\alpha=\frac{1 \pm \lambda}{2} . \tag{40}
\end{equation*}
$$

Thus one sees that the $\alpha \leftrightarrow 1-\alpha$ involutive symmetry in the identities (24) amounts to changing the sign of $\lambda: \quad \lambda \leftrightarrow-\lambda$. The value $\lambda=1$ (associated with the "physical" correlation functions) corresponds to the two values $\alpha=0$ and $\alpha=1$ for which the four factors $f_{n}$ become polynomials of $\tilde{E}$ and $\tilde{K}$ (not necessarily of the same degree see for instance (33), (34)). The value $\lambda=0$ (associated with the algebraic function $\left.C(0,5 ; 0)=(1-t)^{1 / 4}\right)$ corresponds to the value $\alpha=1 / 2$.

Recalling the usual parametrization [8, [14] of the parameter $\lambda$, namely $\lambda=$ $\cos (u)$, and the trigonometric identity

$$
\begin{equation*}
\cos (u)=2 \cos (u / 2)^{2}-1 \tag{41}
\end{equation*}
$$

we see that the parameter $\alpha$ is naturally parameterized as

$$
\begin{equation*}
\alpha=\cos (u / 2)^{2}, \tag{42}
\end{equation*}
$$

the $\alpha \leftrightarrow 1-\alpha$ involutive symmetry in the identities (24) corresponding to the parametrization

$$
\begin{equation*}
1-\alpha=1-\cos (u / 2)^{2}=\sin (u / 2)^{2} \tag{43}
\end{equation*}
$$

which amounts to changing $u$ into $u \rightarrow u+\pi$ in (42), a transformation that does not change $\lambda^{2}=\cos (u)^{2}$.

### 3.2. The algebraic $\alpha=1 / 2$ case

One thus sees that the (involutive) symmetry $\alpha \leftrightarrow 1-\alpha$ singles out $\alpha=1 / 2$. Along this line, note that, for $\alpha=1 / 2$, these $\alpha$-extensions (20), (21) become algebraic functions. One actually has:

$$
\begin{align*}
f_{1}\left(\frac{1}{2}\right) & =\frac{3}{2} \cdot t \cdot(1-t)^{1 / 16} \cdot\left(\frac{1+(1-t)^{1 / 2}}{2}\right)^{5 / 4}  \tag{44}\\
= & \frac{3}{2} t-\frac{9}{16} t^{2}-\frac{15}{128} t^{3}-\frac{15}{256} t^{4}-\frac{1215}{32768} t^{5}-\frac{6903}{262144} t^{6}+\cdots \\
f_{2}\left(\frac{1}{2}\right) & =\frac{3}{2} \cdot t \cdot(1-t)^{1 / 16} \cdot(1-t)^{-1 / 4} \cdot\left(\frac{1+(1-t)^{1 / 2}}{2}\right)^{5 / 4}  \tag{45}\\
& =\frac{3}{2} t-\frac{3}{16} t^{2}-\frac{3}{128} t^{3}+\frac{225}{32768} t^{5}+\frac{2451}{262144} t^{6}+\cdots
\end{align*}
$$

The $\alpha$-extensions (22), (23) for $f_{3}(\alpha)$ and $f_{4}(\alpha)$ also become algebraic functions:

$$
\begin{gather*}
f_{3}\left(\frac{1}{2}\right)=-\frac{3}{8} t^{2}-\frac{3}{32} t^{3}-\frac{45}{1024} t^{4}-\frac{105}{4096} t^{5}-\frac{4395}{262144} t^{6}+\cdots \\
=-\frac{3}{8} \cdot t^{2} \cdot(1-t)^{1 / 16} \cdot\left(\frac{1+(1-t)^{1 / 2}}{2}\right)^{-3 / 4} \cdot\left(\frac{\left(1+t^{1 / 2}\right)^{1 / 2}-\left(1-t^{1 / 2}\right)^{1 / 2}}{t^{1 / 2}}\right) \\
=-\frac{3}{8} \cdot t^{2} \cdot(1-t)^{1 / 16} \cdot\left(\frac{1+(1-t)^{1 / 2}}{2}\right)^{-3 / 4} \cdot\left(2 \cdot \frac{\left(1-(1-t)^{1 / 2}\right)}{t}\right)^{1 / 2},  \tag{46}\\
\begin{aligned}
f_{4}\left(\frac{1}{2}\right)= & -\frac{3}{8} t^{2}-\frac{3}{16} t^{3}-\frac{129}{1024} t^{4}-\frac{195}{2048} t^{5}-\frac{20115}{262144} t^{6}+\cdots \\
= & -\frac{3}{8} \cdot t^{2} \cdot(1-t)^{1 / 16} \cdot(1-t)^{-1 / 4} \cdot\left(\frac{1+(1-t)^{1 / 2}}{2}\right)^{-3 / 4} \\
& \times\left(\frac{\left(1+t^{1 / 2}\right)^{1 / 2}-\left(1-t^{1 / 2}\right)^{1 / 2}}{t^{1 / 2}}\right) \\
= & -\frac{3}{8} \cdot t^{2} \cdot(1-t)^{1 / 16} \cdot(1-t)^{-1 / 4} \cdot\left(\frac{1+(1-t)^{1 / 2}}{2}\right)^{-3 / 4} \\
& \times\left(2 \cdot \frac{\left(1-(1-t)^{1 / 2}\right)}{t}\right)^{1 / 2}
\end{aligned}
\end{gather*}
$$

One verifies easily that

$$
\begin{align*}
& f_{1}\left(\frac{1}{2}\right) \cdot f_{3}\left(\frac{1}{2}\right)=(1-t)^{1 / 2} \cdot f_{2}\left(\frac{1}{2}\right) \cdot f_{4}\left(\frac{1}{2}\right)=-\frac{9}{16} \cdot t^{3} \cdot(1-t)^{1 / 8}  \tag{48}\\
& f_{1}\left(\frac{1}{2}\right) \cdot f_{4}\left(\frac{1}{2}\right)=f_{2}\left(\frac{1}{2}\right) \cdot f_{3}\left(\frac{1}{2}\right)=-\frac{9}{16} \cdot t^{3} \cdot(1-t)^{-1 / 8} \tag{49}
\end{align*}
$$

in agreement with the identities (25) and (26).
Do note that $f_{1}(\alpha)$ and $(1-t)^{1 / 4} \cdot f_{2}(\alpha)$, but also $t^{1 / 4} \cdot f_{3}(\alpha)$ and also $t^{1 / 4} \cdot(1-$ $t)^{1 / 4} \cdot f_{4}(\alpha)$, verify the same Okamoto sigma-form of Painlevé VI (independently of $\alpha$ ). Note that the previous algebraic function solutions (44) and (45) are actually such that $f_{1}\left(\frac{1}{2}\right)$ and $(1-t)^{1 / 4} \cdot f_{2}\left(\frac{1}{2}\right)$ are not only solutions of the same non-linear ODE but are actually the same algebraic function $f_{1}\left(\frac{1}{2}\right)=(1-t)^{1 / 4} \cdot f_{2}\left(\frac{1}{2}\right)$. Similarly (46) and (47) are actually such that $f_{3}\left(\frac{1}{2}\right)$ and $(1-t)^{1 / 4} \cdot f_{4}\left(\frac{1}{2}\right)$ are not only solutions of the same non-linear ODE but are actually the same algebraic function $f_{3}\left(\frac{1}{2}\right)=(1-t)^{1 / 4} \cdot f_{4}\left(\frac{1}{2}\right)$. For $\alpha=1 / 2$ the corresponding $\lambda$ deduced from (39) is $\lambda=0$ and the four product
formula (38) becomes, with the previous exact algebraic expressions (44), (45), (46) and (47) (and after simplifications):

$$
\begin{align*}
& C(0,5 ; 0)=\frac{256}{81} \cdot \frac{(1-t)^{1 / 2}}{t^{6}} \cdot f_{1}\left(\frac{1}{2}\right) \cdot f_{2}\left(\frac{1}{2}\right) \cdot f_{3}\left(\frac{1}{2}\right) \cdot f_{4}\left(\frac{1}{2}\right)=(1-t)^{1 / 4} \\
& \quad=\quad 1-\frac{1}{4} t-\frac{3}{32} t^{2}-\frac{7}{128} t^{3}-\frac{77}{2048} t^{4}-\frac{231}{8192} t^{5}-\frac{1463}{65536} t^{6}+\cdots \tag{50}
\end{align*}
$$

in agreement with the $\lambda=0$ evaluation of the form factor expansion (35). Note that, conversely, the identity (50) can be used to find the exact expressions of the products $f_{1} f_{4}$ and $f_{1} f_{3}$ evaluated at $\alpha=1 / 2$ (see (48) and (49)), when the exact expressions on the $f_{n}$ 's, $n=1,2,3,4$, are much more involved (see (44), (45), (46), (47)).

Remark: All these calculations are not specific of $N=5$. Similar calculations can be performed for other values of $N$. Since these calculations become more and more involved, they will not be detailed here. Let us just give the expressionsit of $f_{1}$ for different (odd) values of $N$, in terms of the complete elliptic integrals of the first and second kind $\tilde{K}$ and $\tilde{E}$.

For $N=5,7,9$, the $f_{1}(N)$ solutions read respectively:

$$
\begin{align*}
f_{1}(N=5)= & (2 t-1) \cdot \tilde{E}+(1-t) \cdot \tilde{K}  \tag{51}\\
f_{1}(N=7)= & -(3 t+4) \cdot(t-1)^{2} \cdot \tilde{K}^{2}+2(t-1) \cdot\left(3 t^{2}-7 t-4\right) \cdot \tilde{E} \tilde{K} \\
& +\left(11 t^{2}-11 t-4\right) \cdot \tilde{E}^{2},  \tag{52}\\
f_{1}(N=9)= & \left(8 t^{2}-47 t+12\right) \cdot(t-1)^{2} \cdot \tilde{K}^{2} \\
& -2 \cdot(t-1) \cdot\left(16 t^{3}-63 t^{2}+83 t-12\right) \cdot \tilde{E} \tilde{K} \\
& +\left(32 t^{4}-64 t^{3}+151 t^{2}-119 t+12\right) \cdot \tilde{E}^{2}, \tag{53}
\end{align*}
$$

We can verify for $N=5,9,13, \cdots$ that the factor $f_{1}(N)$ expands as

$$
\begin{equation*}
f_{1}(N)=\lambda_{N} \cdot t^{(N-1)^{2} / 16}+\cdots \tag{54}
\end{equation*}
$$

when, for $N=7,11,15, \cdots$ the factor $f_{1}(N)$ has the expansion:

$$
\begin{equation*}
f_{1}(N)=\mu_{N} \cdot t^{(N+1)^{2} / 16}+\cdots \tag{55}
\end{equation*}
$$

### 3.3. Form-factor deformation around the algebraic function $f_{1}(1 / 2)$

Introducing a form-factor $\beta$-deformation around the algebraic function (44) ( $\beta$ is the deformation parameter around $\alpha=1 / 2$ )

$$
\begin{align*}
f_{1}\left(\frac{1}{2}+\beta\right) & = \\
& \frac{3}{2} \cdot t \cdot(1-t)^{1 / 16} \cdot\left(\frac{1+(1-t)^{1 / 2}}{2}\right)^{5 / 4}+\beta \cdot G(t)+\cdots \tag{56}
\end{align*}
$$

and inserting (56) in the non-linear ODE verified by (56), one gets an order-three linear differential operator for the first coefficient $G(t)$.
$\dagger$ These expressions can be compared with expressions (E.2) and (E.13) in appendix E of 8 but with a different normalization (E.1).

This order-three linear differential operator has the following solution:

$$
\begin{align*}
G(t) & =\frac{t^{2}}{64} \cdot(1-t)^{1 / 16} \cdot\left(\frac{1+(1-t)^{1 / 2}}{2}\right)^{1 / 4} \cdot P_{E, K}  \tag{57}\\
& =-\frac{15}{1024} \cdot t^{4}-\frac{135}{8192} \cdot t^{5}-\frac{513}{32768} \cdot t^{6}-\frac{7497}{524288} \cdot t^{7}-\frac{434295}{33554432} \cdot t^{8}+\cdots
\end{align*}
$$

where $P_{E, K}$ is a polynomial in $\tilde{E}$ and $\tilde{K}$ :

$$
\begin{align*}
& P_{E, K}=  \tag{58}\\
& \quad\left(t-4+12 \cdot(1-t)^{1 / 2}\right) \cdot{ }_{2} F_{1}\left(\left[\frac{3}{2}, \frac{3}{2}\right],[3], t\right)-8 \cdot{ }_{2} F_{1}\left(\left[\frac{1}{2}, \frac{1}{2}\right],[2], t\right)= \\
& -8 \cdot \frac{12 \cdot(t-2) \cdot(1-t)^{1 / 2}+3 t^{2}-8 t+8}{t^{2}} \cdot \tilde{K}-32 \cdot \frac{t-2+6 \cdot(1-t)^{1 / 2}}{t^{2}} \cdot \tilde{E} .
\end{align*}
$$

As far as the log-derivative with respect to $t$ is concerned, one gets:

$$
\begin{align*}
& t \cdot(t-1) \cdot \frac{d}{d t} \ln \left(f_{1}\left(\frac{1}{2}+\beta\right)\right)=\frac{10 \cdot(1-t)^{1 / 2}+27 t-26}{16}  \tag{59}\\
&-\frac{\beta}{96} \cdot\left(t \cdot(1-t)^{1 / 2} \cdot P_{E, K}+2 \cdot t \cdot(1-t) \cdot\left(1-(1-t)^{1 / 2}\right) \cdot \frac{d P_{E, K}}{d t}\right)+\cdots
\end{align*}
$$

where the first deformation term is also a polynomial in $\tilde{E}$ and $\tilde{K}$.
4. $\alpha$-extensions of the two factors $F_{1}, F_{2}$ for $C(2,5)$

The low-temperature correlation functions $C(M, N)$, at $\nu=-k$, with $M<N$, $M+N$ odd, $M$ even but different from 0 , factor into the product of, not four terms, but only two terms:

$$
\begin{equation*}
C(M, N)=\rho \cdot(1-t)^{1 / 2} \cdot t^{-\left(N^{2}-1\right) / 4} \cdot F_{1}(M, N) \cdot F_{2}(M, N) \tag{60}
\end{equation*}
$$

For instance for $M=2$ and $N=5$ one has

$$
\begin{equation*}
C(2,5)=\frac{256}{2025} \cdot \frac{(1-t)^{1 / 2}}{t^{6}} \cdot F_{1}(2,5) \cdot F_{2}(2,5) \tag{61}
\end{equation*}
$$

where

$$
\begin{array}{rl}
F_{1}(2,5)=2 & 2 \cdot(1-t) \cdot(2 t+1) \cdot \tilde{K}^{2}+\left(7 t^{2}-15 t-4\right) \cdot \tilde{E} \tilde{K} \\
& +\left(2 t^{2}+13 t+2\right) \cdot \tilde{E}^{2} \tag{62}
\end{array}
$$

and:

$$
\begin{align*}
F_{2}(2,5) & =5 \cdot(t-1)^{3} \cdot \tilde{K}^{3}-(11 t-17) \cdot(t-1)^{2} \cdot \tilde{E} \tilde{K}^{2} \\
& +(t-1) \cdot\left(2 t^{2}-33 t+19\right) \cdot \tilde{E}^{2} \tilde{K} \quad+\left(7 t^{2}-22 t+7\right) \cdot \tilde{E}^{3} \tag{63}
\end{align*}
$$

The $\lambda$-extension $C(2,5 ; \lambda)$ corresponds to a form-factor expansion around the algebraic solution $(1-t)^{1 / 4}$ :

$$
\begin{align*}
& C(2,5 ; \lambda)=(1-t)^{1 / 4} \cdot\left(1+\lambda^{2 n} \cdot \sum_{n=1}^{\infty} f_{0,5}^{2 n}\right)  \tag{64}\\
& \quad=1-\frac{t}{4}-\frac{3 t^{2}}{32}-\frac{7 t^{3}}{128}-\frac{77 t^{4}}{2048}-\frac{231 t^{5}}{8192}-\left(\frac{1463}{65536}+\frac{49}{1048576} \cdot \lambda^{2}\right) \cdot t^{6} \\
& \quad-\left(\frac{4807}{262144}+\frac{491}{4194304} \cdot \lambda^{2}\right) \cdot t^{7}-\left(\frac{129789}{8388608}+\frac{205491}{1073741824} \cdot \lambda^{2}\right) \cdot t^{8}+\cdots
\end{align*}
$$

The $\lambda$-extension of (61) can also be seen as a $\mu$-deformation of the correlation function $C(2,5)$, given by the exact expression (61) with (62) and (63), as a polynomial expression in $\tilde{E}$ and $\tilde{K}$ :

$$
\begin{gather*}
C(2,5 ; \lambda)=1-\frac{t}{4}-\frac{3}{32} t^{2}-\frac{7}{128} t^{3}-\frac{77}{2048} t^{4}-\frac{231}{8192} t^{5} \\
-\left(\frac{23457}{1048576}-\frac{49}{1048576} \mu\right) \cdot t^{6}-\left(\frac{7403}{4194304}-\frac{491}{4194304} \mu\right) \cdot t^{7} \\
\quad-\left(\frac{16818483}{1073741824}-\frac{205491}{1073741824} \mu\right) \cdot t^{8}  \tag{65}\\
\quad-\left(\frac{58337917}{4294967296}-\frac{1115389}{4294967296} \mu\right) \cdot t^{9}+\cdots
\end{gather*}
$$

These two series can be seen to identify if one has the following relation between $\lambda$ and $\mu$ :

$$
\begin{equation*}
\lambda^{2}=1-\mu \quad \text { or: } \quad \mu=1-\lambda^{2} \tag{66}
\end{equation*}
$$

The $\alpha$-extension of (62) reads

$$
\begin{gather*}
F_{1}(2,5 ; \alpha)=-\frac{45}{16} t^{3}-\frac{135}{128} t^{4}-\frac{1485}{2048} t^{5}-\left(\frac{4545}{8192}+\frac{315}{8192} \alpha\right) \cdot t^{6} \\
-\left(\frac{58995}{131072}+\frac{17955}{262144} \alpha\right) \cdot t^{7}-\left(\frac{794745}{2097152}+\frac{188055}{2097152} \alpha\right) \cdot t^{8} \\
-\left(\frac{21971565}{67108864}+\frac{876645}{8388608} \alpha\right) \cdot t^{9}+\cdots \tag{67}
\end{gather*}
$$

and the $\alpha$-extension of (63) reads:

$$
\begin{gather*}
F_{2}(2,5 ; \alpha)=-\frac{45}{16} t^{3}+\frac{45}{128} t^{4}+\frac{315}{2048} t^{5}+\left(\frac{315}{4096}+\frac{315}{8192} \alpha\right) \cdot t^{6} \\
+\left(\frac{11655}{262144}+\frac{12915}{262144} \alpha\right) \cdot t^{7}+\left(\frac{14805}{524288}+\frac{106155}{2097152} \alpha\right) \cdot t^{8} \\
+\left(\frac{1285515}{67108864}+\frac{408555}{8388608} \alpha\right) \cdot t^{9}+\cdots \tag{68}
\end{gather*}
$$

One thus verifies that relation (61) can be "lambda-extended"

$$
\begin{equation*}
C(2,5 ; \lambda)=\frac{256}{2025} \cdot \frac{(1-t)^{1 / 2}}{t^{6}} \cdot F_{1}(2,5 ; \alpha) \cdot F_{2}(2,5 ; \alpha) \tag{69}
\end{equation*}
$$

provided:

$$
\begin{equation*}
\mu=4 \cdot \alpha \cdot(1-\alpha) \quad \text { or: } \quad \lambda^{2}=(2 \alpha-1)^{2} \tag{70}
\end{equation*}
$$

Again one verifies the remarkable identities:

$$
\begin{align*}
& F_{2}(2,5 ; \alpha)=(1-t)^{1 / 2} \cdot F_{1}(2,5 ; 1-\alpha) \\
& F_{2}(2,5 ; 1-\alpha)=(1-t)^{1 / 2} \cdot F_{1}(2,5 ; \alpha) \tag{71}
\end{align*}
$$

In particular one has:

$$
\begin{align*}
& F_{1}(2,5 ; 0)= 2 \cdot(1-t) \cdot(2 t+1) \cdot \tilde{K}^{2}+\left(7 t^{2}-15 t-4\right) \cdot \tilde{E} \tilde{K} \\
&+\left(2 t^{2}+13 t+2\right) \cdot \tilde{E}^{2}  \tag{72}\\
& F_{1}(2,5 ; 1)=(1-t)^{-1 / 2} \cdot\left(5 \cdot(t-1)^{3} \cdot \tilde{K}^{3}-(11 t-17) \cdot(t-1)^{2} \cdot \tilde{E} \tilde{K}^{2}\right. \\
&\left.+(t-1) \cdot\left(2 t^{2}-33 t+19\right) \cdot \tilde{E}^{2} \tilde{K}+\left(7 t^{2}-22 t+7\right) \cdot \tilde{E}^{3}\right), \tag{73}
\end{align*}
$$

$$
\begin{align*}
F_{2}(2,5 ; 0)= & 5 \cdot(t-1)^{3} \cdot \tilde{K}^{3}-(11 t-17) \cdot(t-1)^{2} \cdot \tilde{E} \tilde{K}^{2} \\
+ & (t-1) \cdot\left(2 t^{2}-33 t+19\right) \cdot \tilde{E}^{2} \tilde{K}+\left(7 t^{2}-22 t+7\right) \cdot \tilde{E}^{3} \cdot  \tag{74}\\
F_{2}(2,5 ; 1)= & (1-t)^{1 / 2} \cdot\left(2 \cdot(1-t) \cdot(2 t+1) \cdot \tilde{K}^{2}+\left(7 t^{2}-15 t-4\right) \cdot \tilde{E} \tilde{K}\right. \\
& \left.\quad+\left(2 t^{2}+13 t+2\right) \cdot \tilde{E}^{2}\right) . \tag{75}
\end{align*}
$$

The series expansions of the previous exact expressions read:

$$
\begin{align*}
& F_{1}(2,5 ; 0)=-\frac{45}{16} t^{3}-\frac{135}{128} t^{4}-\frac{1485}{2048} t^{5}-\frac{4545}{8192} t^{6}-\frac{58995}{131072} t^{7}+\cdots \\
& F_{1}(2,5 ; 1)=-\frac{45}{16} t^{3}-\frac{135}{128} t^{4}-\frac{1485}{2048} t^{5}-\frac{1215}{2048} t^{6}-\frac{135945}{262144} t^{7}+\cdots  \tag{76}\\
& F_{2}(2,5 ; 0)=-\frac{45}{16} t^{3}+\frac{45}{128} t^{4}+\frac{315}{2048} t^{5}+\frac{315}{4096} t^{6}+\frac{11655}{262144} t^{7}+\cdots \\
& F_{2}(2,5 ; 1)=-\frac{45}{16} t^{3}+\frac{45}{128} t^{4}+\frac{315}{2048} t^{5}+\frac{945}{8192} t^{6}+\frac{12285}{131072} t^{7}+\cdots \tag{77}
\end{align*}
$$

It is worth noticing that (in contrast with the $\lambda$-extension $C(2,5 ; \lambda)$ ), the $F_{n}(2,5 ; \alpha)$ 's have two different values of the parameter $\alpha$ for which these $\alpha$-extensions are Dfinite, being (homogeneous) polynomials in $\tilde{E}$ and $\tilde{K}$. One remarks with that the corresponding polynomials in $\tilde{E}$ and $\tilde{K}$ are not necessarily of the same degree in $\tilde{E}$ and $\tilde{K}$.

Remark: the $\alpha=1 / 2$ algebraic subcase. For $\alpha=1 / 2$ the corresponding $\lambda$ deduced from (70) is $\lambda=0$ and the two product formula (69) becomes

$$
\begin{equation*}
C(2,5 ; 0)=\frac{256}{2025} \cdot \frac{(1-t)^{1 / 2}}{t^{6}} \cdot F_{1}\left(2,5 ; \frac{1}{2}\right) \cdot F_{2}\left(2,5 ; \frac{1}{2}\right)=(1-t)^{1 / 4} \tag{78}
\end{equation*}
$$

in agreement with the expansion (65) evaluated at $\lambda=0$. Using the identity (71) one gets

$$
\begin{equation*}
F_{2}\left(2,5 ; \frac{1}{2}\right)=(1-t)^{1 / 2} \cdot F_{1}\left(2,5 ; \frac{1}{2}\right) \tag{79}
\end{equation*}
$$

which enables to write (78) as:

$$
\begin{equation*}
C(2,5 ; 0)=\frac{256}{2025} \cdot \frac{1}{t^{6}} \cdot\left(F_{2}\left(2,5 ; \frac{1}{2}\right)\right)^{2}=(1-t)^{1 / 4} \tag{80}
\end{equation*}
$$

from which one deduces

$$
\begin{align*}
& F_{2}\left(2,5 ; \frac{1}{2}\right)=-\frac{45}{16} \cdot t^{3} \cdot(1-t)^{1 / 8} \\
& \quad=-\frac{45}{16} t^{3}+\frac{45}{128} t^{4}+\frac{315}{2048} t^{5}+\frac{1575}{16384} t^{6}+\frac{36225}{524288} t^{7}+\cdots \tag{81}
\end{align*}
$$

or:

$$
\begin{align*}
& F_{1}\left(2,5 ; \frac{1}{2}\right)=-\frac{45}{16} \cdot t^{3} \cdot(1-t)^{-3 / 8} \\
& \quad=-\frac{45}{16} t^{3}-\frac{135}{128} t^{4}-\frac{1485}{2048} t^{5}-\frac{9405}{16384} t^{6}-\frac{253935}{524288} t^{7}+\cdots \tag{82}
\end{align*}
$$

### 4.1. Form factor deformation around the algebraic function $F_{1}\left(2,5 ; \frac{1}{2}\right)$

Introducing a form-factor $\beta$-deformation around the algebraic function (82) ( $\beta$ is the deformation parameter around $\alpha=1 / 2$ )

$$
\begin{equation*}
F_{1}\left(2,5 ; \frac{1}{2}+\beta\right)=-\frac{45}{16} \cdot t^{3} \cdot(1-t)^{-3 / 8}+\beta \cdot G(t)+\cdots \tag{83}
\end{equation*}
$$

and inserting (83) in the non-linear ODE verified by (83), one gets an order-three linear differential operator which is the direct sum of an order-one linear differential operator and an order-two linear differential operator, yielding the following exact expression for $G(t)$ in (83):

$$
\begin{align*}
& G(t)=-\frac{45}{16} \cdot t^{3} \cdot(1-t)^{-3 / 8}-\frac{9}{16} \cdot(1-t)^{-3 / 8} \cdot P_{E, K}  \tag{84}\\
& \quad=-\frac{315}{8192} \cdot t^{6}-\frac{17955}{262144} \cdot t^{7}-\frac{188055}{2097152} \cdot t^{8}-\frac{876645}{8388608} \cdot t^{9}-\frac{1929015}{16777216} \cdot t^{10}+\cdots
\end{align*}
$$

where $P_{E, K}$ is a polynomial in $\tilde{E}$ and $\tilde{K}$ :

$$
\begin{align*}
P_{E, K}= & 4 \cdot t^{2} \cdot(t-1) \cdot\left(t^{2}-6 t+16\right) \cdot \frac{d \tilde{K}}{d t}+t^{2} \cdot\left(2 t^{2}-13 t+16\right) \cdot K \\
& =t \cdot\left(t^{2}-28 t+32\right) \cdot \tilde{K}-2 \cdot\left(t^{2}-6 t+16\right) \cdot \tilde{E} \tag{85}
\end{align*}
$$

As far as the log-derivative with respect to $t$ is concerned, one gets:

$$
\begin{align*}
t \cdot(t-1) \cdot & \frac{d}{d t} \ln \left(F_{1}\left(2,5 ; \frac{1}{2}+\beta\right)\right)=-3+\frac{21}{8} \cdot t \\
& +\beta \cdot \frac{t-1}{5 t^{3}} \cdot\left(t \cdot \frac{d P_{E, K}}{d t}-3 \cdot P_{E, K}\right)+\cdots \tag{86}
\end{align*}
$$

where the first deformation term is also polynomial in $\tilde{E}$ and $\tilde{K}$.

## 5. Comments and speculations on the lambda-extensions of the two-point correlation functions.

The previous sections provide an illustration of nice involutive symmetries of $\alpha$ extension solutions of Painlevé-like non-linear ODEs (see (24)). Furthermore, recalling (31), (32), (46) and (33), (34), (47), namely

$$
\begin{align*}
& f_{3}(0)=(t-2) \cdot \tilde{E}+2 \cdot(1-t) \cdot \tilde{K} \\
& f_{3}(1)=(1-t)^{-1 / 4} \cdot\left(3 \tilde{E}^{2}+2 \cdot(t-2) \cdot \tilde{E} \tilde{K}+(1-t) \cdot \tilde{K}^{2}\right)  \tag{87}\\
& f_{3}\left(\frac{1}{2}\right)=-\frac{3}{8} \cdot t^{2} \cdot(1-t)^{1 / 16} \cdot\left(\frac{1+(1-t)^{1 / 2}}{2}\right)^{-3 / 4} \cdot\left(2 \cdot \frac{\left(1-(1-t)^{1 / 2}\right)}{t}\right)^{1 / 2}
\end{align*}
$$

and

$$
\begin{align*}
f_{4}(0)= & 3 \tilde{E}^{2}+2 \cdot(t-2) \cdot \tilde{E} \tilde{K}+(1-t) \cdot \tilde{K}^{2} \\
f_{4}(1)= & (1-t)^{1 / 4} \cdot((t-2) \cdot \tilde{E}+2 \cdot(1-t) \cdot \tilde{K})  \tag{88}\\
f_{4}\left(\frac{1}{2}\right)=-\frac{3}{8} \cdot t^{2} \cdot & (1-t)^{1 / 16} \cdot(1-t)^{-1 / 4} \cdot\left(\frac{1+(1-t)^{1 / 2}}{2}\right)^{-3 / 4} \\
& \times\left(2 \cdot \frac{\left(1-(1-t)^{1 / 2}\right)}{t}\right)^{1 / 2}
\end{align*}
$$

we see that the $\alpha$-extension $f_{3}(\alpha)$ (resp. $f_{4}(\alpha)$ ) has three different values of the parameter $\alpha$ for which the corresponding $\alpha$-extensions are D-finite being (homogeneous) polynomials in $\tilde{E}$ and $\tilde{K}$ of different degree in $\tilde{E}$ and $\tilde{K}$. It is straightforward to see that $f_{3}(\alpha)$ (resp. $f_{4}(\alpha)$ ) is not a linear interpolation of these three D-finite expressions. For generic values of $\alpha, f_{3}(\alpha)$ (resp. $f_{4}(\alpha)$ ) is not D-finit $\ddagger$, it is differentially algebraic [11, 12, 33, being solution of a Painlevé-like non-linear ODE. Let us now display several remarkable properties of such lambda-extensions.

### 5.1. Other remarkable features of the lambda-extensions of the two-point correlation functions.

In fact $\alpha=1 / 2$ is not the only value of $\alpha$ for which $f_{3}(\alpha)$ (resp. $f_{4}(\alpha)$ ) becomes an algebraic function. One has an infinite number of (algebraic) values of $\alpha$ for which $f_{3}(\alpha)$ (resp. $f_{4}(\alpha)$ ) becomes an algebraic function. This phenomenon is illustrated in detail in [32] in the case of the lambda-extension of the diagonald correlation function $C(1,1)$, but one has similar results for other non-diagonal twopoint correlation functions (at $\nu=-k$ ), or for factors of the correlation functions like the $f_{i}(\alpha)$ 's.

For pedagogical reasons we restrict our analysis to the low-temperature two-point correlation function $C(1,1)$ and its lambda extension. For instance, the form factor expansion of the lambda extension of this low-temperature correlation function reads

$$
\begin{equation*}
C_{-}(1,1 ; \lambda)=(1-t)^{1 / 4} \cdot\left(1+\sum_{n=1}^{\infty} \lambda^{2 n} \cdot f_{1,1}^{(2 n)}\right) \tag{89}
\end{equation*}
$$

where the first form factors read:

$$
\begin{align*}
f_{1,1}^{(2)}= & \frac{1}{2} \cdot\left(1-3 E K-(t-2) \cdot K^{2}\right)  \tag{90}\\
f_{1,1}^{(4)}= & \frac{1}{24} \cdot\left(9-30 \tilde{E} \tilde{K}-10 \cdot(t-2) \cdot \tilde{K}^{2}\right. \\
& \left.+\left(t^{2}-6 t+6\right) \cdot \tilde{K}^{4}+15 \tilde{E}^{2} \tilde{K}^{2}+10 \cdot(t-2) \cdot \tilde{E} \tilde{K}^{3}\right) \tag{91}
\end{align*}
$$

For $\lambda=1$ we must recover, from (89), the well-known expression of the lowtemperature two-point correlation function $C(1,1)=\tilde{E}$ :

$$
\begin{align*}
C_{-}(1,1 ; 1) & =E=1-\frac{1}{4} \cdot t-\frac{3}{64} \cdot t^{2}-\frac{5}{256} \cdot t^{3}-\frac{175}{16384} \cdot t^{4}+\cdots \\
& =(1-t)^{1 / 4} \cdot\left(1+\sum_{n=1}^{\infty} f_{1,1}^{(2 n)}\right) \tag{92}
\end{align*}
$$

which corresponds to write the ratio $\tilde{E} /(1-t)^{1 / 4}$ as an infinite sum of polynomial expressions of $\tilde{E}$ and $\tilde{K}$, thus yielding a non-trivial infinite sum identity on the complete elliptic integrals $\tilde{E}$ and $\tilde{K}$.

Since all these lambda extensions are power series in $t$, we can also try to get, order by order, the series expansion of $C_{-}(1,1 ; \lambda)$ from the corresponding nonlinear ODE (see (104) below). Recalling [14] the form factor expansion (89), we can either see the series expansion in $t$ as a deformation of the simple algebraic function

[^1]$(1-t)^{1 / 4}$, or more naturally, see the series expansion of the lambda-extension of the low-temperature two-point correlation function $C_{-}(1,1 ; \lambda)$ as a deformation of the exact expression $C_{-}(1,1)=\tilde{E}$ (here $M$ denotes here a difference to $\lambda^{2}=1$, namely $\left.M=4 \cdot\left(1-\lambda^{2}\right)\right)$ :
\[

$$
\begin{align*}
C_{-}(1,1 ; \lambda) & =C_{M}(1,1 ; M) \\
& =\tilde{E}+M \cdot g_{1}(t)+M^{2} \cdot g_{2}(t)+M^{3} \cdot g_{3}(t) \quad+\cdots \tag{93}
\end{align*}
$$
\]

All the $g_{n}(t)$ 's in (93) are also [32] polynomialst in $\tilde{E}$ and $\tilde{K}$. For instance $g_{1}(t)$ in (95) reads:

$$
\begin{equation*}
g_{1}(t)=\frac{1}{24} \cdot \tilde{E}-\frac{1}{8} \cdot \tilde{K} \tilde{E}^{2}-\frac{t-1}{12} \cdot \tilde{K}^{3} \tag{94}
\end{equation*}
$$

Using the sigma-form of Painlevé VI equation (104) one can find that this expansion (93) reads as a series expansion in the variable $t$ :

$$
\begin{gather*}
C_{M}(1,1 ; M)=1-\frac{1}{4} \cdot t-\left(\frac{3}{64}+\frac{3}{256} \cdot M\right) \cdot t^{2}-\left(\frac{5}{256}+\frac{9}{1024} \cdot M\right) \cdot t^{3} \\
-\left(\frac{175}{16384}+\frac{441}{65536} \cdot M\right) \cdot t^{4}-\left(\frac{441}{65536}+\frac{1407}{262144} \cdot M\right) \cdot t^{5} \\
-\left(\frac{4851}{1048576}+\frac{9281}{2097152} \cdot M-\frac{5}{16777216} \cdot M^{2}\right) \cdot t^{6}+\cdots \tag{95}
\end{gather*}
$$

### 5.1.1. Deformation around an algebraic subcase.

Recalling that one finds [32] that (95) is actually, for $M=2$, the series expansion of an algebraic function (see (97) below), one can try to write the series (95) as a deformation of this $M=2$ algebraic function (97)

$$
\begin{equation*}
C_{\rho}(1,1 ; \rho)=G_{0}(t)+\rho \cdot G_{1}(t)+\rho^{2} \cdot G_{2}(t)+\cdots \tag{96}
\end{equation*}
$$

where

$$
\begin{equation*}
G_{0}(t)=(1-t)^{1 / 16} \cdot\left(\frac{1+(1-t)^{1 / 2}}{2}\right)^{3 / 4} \tag{97}
\end{equation*}
$$

and where $\rho=M-2$. Again one can ask whether the $G_{n}(t)$ 's in (96) are D-finite, and, again, polynomials in the complete elliptic integrals $\tilde{E}$ and $\tilde{K}$. This is actually the case. One finds that (96) can be written as

$$
\begin{gather*}
\frac{C_{\rho}(1,1 ; \rho)}{G_{0}(t)}=1+\rho \cdot\left(\frac{1}{4} \cdot S_{2}-\frac{1}{4}\right)+\rho^{2} \cdot\left(\frac{1}{32} \cdot S_{3}-\frac{1}{16} \cdot S_{2}+\frac{3}{32}\right) \\
\quad+\rho^{3} \cdot\left(\frac{1}{384} \cdot S_{4}-\frac{1}{128} \cdot S_{3}+\frac{13}{384} \cdot S_{2}-\frac{5}{128}\right)+\cdots \tag{98}
\end{gather*}
$$

where:

$$
\begin{aligned}
S_{2}= & \frac{2}{t} \cdot\left(1-(1-t)^{1 / 2}\right) \cdot \tilde{E}-\frac{1}{2 t} \cdot\left((t-4) \cdot(1-t)^{1 / 2}-(3 t-4)\right) \cdot \tilde{K} \\
S_{3}= & \frac{1}{4} \cdot\left(6 \cdot(1-t)^{1 / 2}-(t-2)\right) \cdot \tilde{K}^{2}-3 \tilde{E} \tilde{K} \\
S_{4}= & \frac{3}{t} \cdot\left((t-4) \cdot(1-t)^{1 / 2}-(3 t-4)\right) \cdot \tilde{E} \tilde{K}^{2}-\frac{6}{t} \cdot\left(1-(1-t)^{1 / 2}\right) \cdot \tilde{E}^{2} \tilde{K} \\
& \quad+\frac{1}{8 t} \cdot\left(\left(t^{2}-28 t+48\right) \cdot(1-t)^{1 / 2}-\left(21 t^{2}-68 t+48\right)\right) \cdot \tilde{K}^{3}
\end{aligned}
$$

$\dagger \dagger$ This cannot be deduced straightforwardly from an identification of two representations (95) and (96) of the lambda extension $C_{-}(1,1 ; \lambda)$. This identification yields an infinite number of (infinite sum) non-trivial identities on $\tilde{E}$ and $\tilde{K}$.

We thus see the same phenomenon as the one sketched in section (3.3) for the $\alpha$ extension $f_{1}(\alpha)$ and section (4.1) for the $\alpha$-extension $F_{1}(2,5 ; \alpha)$, seen as deformations of algebraic function subcases.

Remark: All these $g_{n}(t)$ 's or $G_{n}(t)$ 's are globally bounded serie $\ddagger$ 34. This is a consequence of the fact that they are polynomial expressions in $\tilde{E}$ and $\tilde{K}$ : they are not only D-finite, they can actually be seen to be diagonals of rational functions [34. We have actually seen, so many times in physics, and in particular in the two-dimensional Ising model, the emergence of globally bounded series as a consequence of the frequent occurrence of diagonals of rational functions [34, 35] (or $n$-fold integrals [36]). In contrast the lambda extension $C_{-}(1,1 ; \lambda)$ which is an infinite sum of globally bounded series is, at first sight, a differentially algebraic function which has no reason to correspond to a globally bounded series.

### 5.2. Arithmetic properties of the lambda-extensions and globally bounded series.

Let us consider the series expansion (95) for values of the parameter $M \neq 0$ not yielding the previous algebraic function series (i.e. $M \neq 4 \cdot \sin ^{2}(\pi m / n)$ where $m$ and $n$ are integers).

Let us change $t$ into $16 t$ in the series expansion (95). One gets the following expansion:

$$
\begin{align*}
& 1-4 t-(12+3 M) \cdot t^{2}-(80+36 M) \cdot t^{3}-(700+441 M) \cdot t^{4} \\
& -(7056+5628 M) \cdot t^{5}-\left(77616+74248 M-5 M^{2}\right) \cdot t^{6} \\
& -\left(906048+1004960 M-220 M^{2}\right) \cdot t^{7}-\left(11042460+13877397 M-6255 M^{2}\right) \cdot t^{8} \\
& -\left(139053200+194712812 M-146500 M^{2}\right) \cdot t^{9} \\
& -\left(1796567344+2767635832 M-3079025 M^{2}\right) \cdot t^{10}+\cdots \tag{99}
\end{align*}
$$

For integer values of $M$ one sees, very clearly, that the series (99) becomes a differentially algebrai $\ddagger$ series with integer coefficients. One thus has a first example of an infinite number of differentially algebraic series with integer coefficients. As far as integer values of $M$ are concerned we have seen [32] that the lambda extension $C_{-}(1,1 ; \lambda)$ is a simple algebraic function for $M=2,4 \underset{\sim}{4}$ and slightly more involved algebraic functions for $M=1,3$, and corresponds to $\tilde{E}$ for $M=0$. These series (99) are, at first sight, differentially algebraic [11]: is it possible that such series could become D-finite for selected values integer of $M$ different from $M=0,1,2,3,4$ ?

In section (4.1) of [32] we give some strong argument to discard, at least for $M=5$, the possibility that the series expansion (95) (or the series expansion (99)) could be D-finite. It is differentially algebraic.

More generally, one can see that the series expansion (95) (or the series expansion (99)) is a globally bounded series when $M$ is any rational number. One thus generalizes the quite puzzling result that an infinite number of (at first sight ...) differentially algebraic series can be globally bounded series.

Remark: Quite often we see the emergence of globally bounded series 34 as solutions of D-finite linear differential operators, and more specifically as diagonals of rational functions [34, 35] (this is related to the so-called Christol's conjecture 37]).
$\dagger$ A series with rational coefficients and non-zero radius of convergence is a globally bounded series 34] if it can be recast into a series with integer coefficients with one rescaling $t \rightarrow N t$ where $N$ is an integer.
$\ddagger$ They are solutions of a non-linear ODE, the sigma-form of Painlevé VI.

The emergence of globally bounded series that are not D-finite (not diagonals of rational functions) is more puzzling. It can be tempting to imagine that such differentially algebraic globally bounded situations could correspond to particular ratios of D-finite functions, namely ratios of diagonals of rational functions (or even rational functions of diagonals), or even composition of diagonal of rational functions. Our prejudice is that this is not the case, but discarding these simple scenarii is extremely difficult.

## 6. More non-linear ODEs of the Painlevé type and more $\lambda$-extensions.

In 38 V.V. Mangazeev and A. J. Guttmann derived the following Toda-type recurrence relation for the $\lambda$-extension $C(N, N ; \lambda)$ of the diagonal correlation functions of the square Ising model (see equation (6) in [38):

$$
\begin{equation*}
t \cdot \frac{d^{2}}{d t^{2}} \ln \left(C_{N}\right)+\frac{d}{d t} \ln \left(C_{N}\right)+\frac{N^{2}}{1-t^{2}}=\frac{N^{2}-1 / 4}{1-t^{2}} \cdot \frac{C_{N-1} \cdot C_{N+1}}{C_{N}^{2}} \tag{100}
\end{equation*}
$$

where $C_{N}$ denotes the $\lambda$-extensions of the low (resp. high) diagonal correlation functions $C_{N}=C(N, N)$. Introducing the ratio

$$
\begin{equation*}
R_{N}=\frac{C_{N-1} \cdot C_{N+1}}{C_{N}^{2}} \quad \text { or: } \quad P_{N}=\frac{N^{2}-1 / 4}{1-t^{2}} \cdot \frac{C_{N-1} \cdot C_{N+1}}{C_{N}^{2}} \tag{101}
\end{equation*}
$$

one can easily deduce from (100) (together with the same relation (100) where $N$ is changed into $N-1$ and $N+1$ ) other relations like:

$$
\begin{align*}
& t \cdot \frac{d}{d t}\left(t \cdot \frac{d \ln \left(R_{N}\right)}{d t}\right)+\frac{2}{(1-t)^{2}}  \tag{102}\\
& \quad=\frac{(N-1)^{2}-1 / 4}{1-t^{2}} \cdot R_{N-1}+\frac{(N+1)^{2}-1 / 4}{1-t^{2}} \cdot R_{N+1}-2 \cdot \frac{N^{2}-1 / 4}{1-t^{2}} \cdot R_{N}
\end{align*}
$$

or:

$$
\begin{equation*}
\left(t \cdot \frac{d}{d t}\right)^{2} \ln \left(P_{N}\right)+\frac{2}{1-t}=P_{N-1}+P_{N+1}-2 P_{N} \tag{103}
\end{equation*}
$$

Let us now consider, for instance, the low-temperature $T<T_{c}$ diagonal correlation functions. One knows that they verify the sigma-form of Painlevé VI equation

$$
\begin{align*}
& \left(t \cdot(t-1) \cdot \frac{d^{2} \sigma}{d t^{2}}\right)^{2}  \tag{104}\\
& \quad=N^{2} \cdot\left((t-1) \cdot \frac{d \sigma}{d t}-\sigma\right)^{2}-4 \cdot \frac{d \sigma}{d t} \cdot\left((t-1) \cdot \frac{d \sigma}{d t}-\sigma-\frac{1}{4}\right) \cdot\left(t \frac{d \sigma}{d t}-\sigma\right)
\end{align*}
$$

with

$$
\begin{equation*}
\sigma=t \cdot(t-1) \cdot \frac{d}{d t} \ln C(N, N)-\frac{t}{4} \tag{105}
\end{equation*}
$$

We can rewrite (100) in terms of $\sigma$ given by (105):

$$
\begin{equation*}
\frac{d}{d t} \ln C_{N}=\frac{\sigma+\frac{t}{4}}{t \cdot(t-1)} \tag{106}
\end{equation*}
$$

TI Let us recall that ratios of D-finite expressions are not (generically) D-finite: they are differentially algebraic [11].

Relation (100) becomes $\mathcal{L}=\mathcal{R}$ where:

$$
\begin{align*}
\mathcal{L} & =t \cdot \frac{d}{d t}\left(\frac{\sigma+\frac{t}{4}}{t \cdot(t-1)}\right)+\frac{\sigma+\frac{t}{4}}{t \cdot(t-1)}+\frac{N^{2}}{1-t^{2}} \\
\mathcal{R} & =\frac{N^{2}-1 / 4}{1-t^{2}} \cdot \frac{C_{N-1} \cdot C_{N+1}}{C_{N}^{2}} \tag{107}
\end{align*}
$$

Let us introduce a new sigma corresponding to the product $C_{N-1} \cdot C_{N+1}$ :

$$
\begin{equation*}
\Sigma=t \cdot(t-1) \cdot \frac{d}{d t} \ln \left(C_{N-1} \cdot C_{N+1}\right) \tag{108}
\end{equation*}
$$

Taking a well-suited log-derivatives the previous relation $\mathcal{L}=\mathcal{R}$ yields:

$$
\begin{equation*}
t \cdot(t-1) \cdot \frac{d}{d t} \ln \mathcal{L}=t \cdot(t-1) \cdot \frac{d}{d t} \ln \mathcal{R} \tag{109}
\end{equation*}
$$

where the RHS of (109) can be written using (105) and (108)

$$
\begin{equation*}
\Sigma-2 \sigma-\frac{5 t}{2} \tag{110}
\end{equation*}
$$

Relation (109) becomes:

$$
\begin{align*}
& 8 \cdot t \cdot(t-1)^{2} \cdot \sigma^{\prime \prime}+4 \cdot(t-1) \cdot(t+4 \sigma) \cdot \sigma^{\prime}-16 \cdot \sigma^{2}+4 \cdot\left(4 N^{2}-1-t\right) \cdot \sigma \\
& \quad+\left(4 N^{2}-1\right) \cdot t-2 \cdot\left(4 N^{2}-1+4 \cdot(t-1) \cdot \sigma^{\prime}-4 \sigma\right) \cdot \Sigma=0 \tag{111}
\end{align*}
$$

We can now use the non-linear ODE (104) to perform some differential algebra eliminations to eliminate $\sigma$ and its derivatives in order to get a non-linear ODE on $\Sigma$. One first eliminates $\sigma^{\prime \prime}$ between (104) and (111), getting a (non-linear) relation between $\sigma, \sigma^{\prime}$ and $\Sigma$. Performing a derivation of this relation one gets a relation between $\sigma, \sigma^{\prime}, \sigma^{\prime \prime}, \Sigma$ and $\Sigma^{\prime}$. Again one eliminates $\sigma^{\prime \prime}$ between this last relation and (111), getting a relation between $\sigma, \sigma^{\prime}, \Sigma$ and $\Sigma^{\prime}$. The elimination of $\sigma^{\prime}$ using a previous relation gives a relation between $\sigma, \Sigma$ and $\Sigma^{\prime}$. A new derivation gives a relation between $\sigma, \Sigma, \Sigma^{\prime}$ and $\Sigma^{\prime \prime}$. Finally eliminating $\sigma$, one gets a non-linear ODE between $\Sigma, \Sigma^{\prime}$ and $\Sigma^{\prime \prime}$. In other words one can obtain a second order non-linear ODE on $\Sigma$, from the Toda-like relation (100) and the sigma-form of Painlevé VI nonlinear ODE (104). This non-linear ODE is too large to be given heret. However, it is worth noticing that, again, this non-linear ODE has one-parameter lambdaextension solution. One may conjecture that this new non-linear ODE has again the (fixed critical point) Painlevé property. This (very large) second order non-linear ODE is not quadratic in the second derivative $\Sigma^{\prime \prime}$, in contrast with Okamoto sigma form of Painlevé VI equation. It is of a much higher degre $\$$. The question of the reduction of this quite large non-linear ODE to some Okamoto sigma-form of Painlevé VI, or more generally to second order non-linear ODE of the Painlevé type [40, is a (challenging) open question. The transformations required to achieve such reduction to the sigma-form of Painlevé VI will correspond to drastic generalization§ of the concept of "folding transformations" 41, 42, 43].

[^2]
### 6.1. Another non-linear $O D E$.

If one tries to obtain, more directly, a non-linear ODE on the product of the two diagonal correlation functions $C(N, N) \cdot C(N+2, N+2)$, one can also consider the sigma-form of Painlevé VI equation (104) together with the definition of sigma (105) and the same equation and definition (104) and (105), but for $N+2$, and obtain by differential algebra eliminations a non-linear ODE on the sum

$$
\begin{align*}
\Sigma & =t \cdot(t-1) \cdot \frac{d}{d t} \ln \left(C_{N} \cdot C_{N+2}\right) \\
& =t \cdot(t-1) \cdot \frac{d}{d t} \ln C(N, N)+t \cdot(t-1) \cdot \frac{d}{d t} \ln C(N+2, N+2) . \tag{112}
\end{align*}
$$

which is essentially the sum of the two previous sigmas (equation (105) for $N$ and for $N+2$ ). Let us recall (see page 344 in [33] and [12]) the results on sums (but also products, compositions, derivatives, integrals, inverses, etc ...) of differentially algebraic functions, showing that these sums are also differentially algebraic functions, and that one also has (see Theorem 2.2 page 345 in [33]) that the order of the nonlinear ODE for such sums is less or equal to the sum of the order of the two non-linear ODEs. In our case (112) one expects the order of the non-linear ODE on $\Sigma$ to be less or equal to $4=2+2$ with a prejudice for the generic upper bound being four.

Comment: We thus have, at first sight, two non-linear ODEs on (112): a very large but second order non-linear ODE obtained by differential algebra eliminations between (104) and (111), and another one, probably also very large but fourth order non-linear ODE. Both equations probably have the fixed critical point Painlevé property. As far as lambda-extensions are concerned, we expect the first one to have one-parameter family of power-series analytic at $t=0$, when we expect twoparameters families of power-series analytic at $t=0$ (the two lambda parameters for $\sigma(N)$ and $\sigma(N+2)$ are, now, independent). Understanding these different non-linear ODEs occurring on products of two-point correlation functions and their corresponding lambda extensions remains a challenging work-in-progress task.

Remark: Quantum XY chain correlations. Along this line, it is worth recalling that the emergence of the product $C_{N-1} \cdot C_{N+1}$, or $C(N, N) \cdot C(N+2, N+2)$, is reminiscent of the product $C(N, N) \cdot C(N+1, N+1)$ which is actually the $x x$ correlation functions of the quantum XY chain in the absence of a magnetic field. Actually, for the xx correlations of the quantum XY chain, one has (see (2.45a) and (2.45b) in Lieb, Schultz and Mattis paper 44) the following relations only valid in the absence of a magnetic field $H=0$ i.e. precisely $\nu=-k$ :

$$
\begin{align*}
& <\sigma_{0}^{x} \sigma_{2 N}^{x}>=C(N, N)^{2}  \tag{113}\\
& <\sigma_{0}^{x} \sigma_{2 N-1}^{x}>=C(N, N) \cdot C(N-1, N-1) \tag{114}
\end{align*}
$$

Again, from the previous results, we have a strong incentive to find the non-linear ODEs for the quantum XY chain correlations (114).

More generally we have a strong incentive to find non-linear ODEs of the Painlevé type for various families of two-point correlation functions like the offdiagonal correlations $C(N, N+1)$ for which N.Witte showed 45] the existence of a Garnier system for such correlations, and, beyond, $C(N, N+2), C(N, N+3), \ldots$ correlationst.

[^3]
## 7. Conclusion.

As underlined in the introduction the two-point correlation functions $C(M, N)$ of the 2D Ising model, at $\nu=-k$, can be seen as D-finite functions solutions of linear differential equations, but also, in the same time, as solutions of non-linear differential equations of the Painleve type. Around $t=0$ the other solutions of the linear differential equations are formal series with logarithms (see [14, 23]). In contrast other solutions of the non-linear differential equations of the Painlevé type are one-parameter families of power series analytic at $t=0$. Such solutions are called lambda-extensions [32]. This paper has tried to provide an illustration of a set of the remarkable properties and structures of such lambda-extensions (resp. $\alpha$-extensions). The study of non-linear ODEs in the most general framework may look hopeless for mathematicians. However, the square Ising model provides a perfect example of the importance of a selected set of non-linear ODEs, namely non-linear ODEs of the Painlevé type 47, and we tried to show that the analysis of some of their solutions, the lambda-extensions, is clearly a powerful way to describe these selected non-linear ODEs in a work-in-progress definition of what could be called the "symmetries" of these non-linear ODEs of the Painleve type.

Although Painlevé equations were introduced from purely mathematical considerations their occurrence in so many domains of physics and theoretical physics is compelling. Let us quote pele mele: particle physics, solid state physics, field theory, lattice statistical mechanics, statistical physics [17], integrable PDE's and their similarity solutions, enumerative combinatorics, Random Matrix Theory 29, 48, even Quantum Gravity [49, the Ising model being, of course, the perfect play ground for these remarkable non-linear ODEs. Unfortunately the compelling evidence of the relevance of these selected non-linear ODEs in physics, is not able to balance the mainstream opinion among pure mathematicians that nothing interesting can be done on non-linear problems and that even the word "non-linear" is meaninglesst. We tried in this paper to show that interesting non-trivial results can be obtained on selected non-linear ODEs.

The exact results sketched in this paper are a strong incentive to get more nonlinear ODEs, for instance on the correlation functions of XY quantum chain in the absence of magnetic field (which corresponds to the product of two Ising two-point Ising correlation functions $C(N, N) \cdot C(N+1, N+1)$, but also on many more two point off-diagonal correlation functions of the 2 D Ising like $C(N, N+1)$, or $C(N, N+2)$, or $C(N, N+3)$.

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[^0]:    ब The non-linear ODEs emerging in such an "integrable" framework are highly selected: they have the (fixed critical) Painlevé property, they have algebraic function solutions, etc ... This is in (strong) contrast with the generic non-linear ODEs for which more numerical analysis (investigation of the qualitative behavior of non linear ODEs, stability and boundness, ...) must be performed (see for instance (3) 4, 5).
    $\ddagger$ A differentially algebraic function 11 12 is a function $f(t)$ solution of a polynomial relation $P\left(t, f(t), f^{\prime}(t), \cdots f^{(n)}(t)\right)=0$, where $f^{(n)}(t)$ denotes the $n$-th derivative of $f(t)$ with respect to $t$.
    $\dagger \dagger$ The symmetries of a differential equation and the symmetries of the solutions of the differential equation are two different concepts.

[^1]:    $\dagger$ In section 4.1 of 32 we provide, not a proof, but arguments strongly suggesting that such lambdaextensions are not generically D-finite.
    I Recall that diagonal correlation functions depend only on $k=\sqrt{t}$. They are independent of the anisotropic parameter $\nu$.

[^2]:    $\dagger$ The non-linear ODE emerges from a resultant that factors in different spurious terms, a polynomial in $\Sigma, \Sigma^{\prime}$ and $\Sigma^{\prime \prime}$ of degree six in $\Sigma^{\prime \prime}$ and another polynomial in $\Sigma, \Sigma^{\prime}$ and $\Sigma^{\prime \prime}$ of degree twelve in $\Sigma^{\prime \prime}$ 。
    $\ddagger$ Along this second order but higher degree line let us recall 39 .
    $\S$ In the simple case of the reduction of a second-order non-Okamoto non linear ODE to an Okamoto sigma form of Painlevé VI equation, equations (26), (28) in section 2 of [8], give some hint of the complexity of such transformations.

[^3]:    - Note that the non-linear ODE for (113) is obviously an Okamoto sigma-form of Painlevé VI equation similar to 105 .
    $\dagger$ The row correlation function $C(0, N)$ is a tau-function of a Garnier system with five finite

[^4]:    singularities, one fixed at the origin: see Corr.1, pg. 7 and Eq.(36), pg. 6 of [46], when $C(N, N+1)$ is more a component of a related isomonodromic system (at least in the description in 45). Preliminary studies for the row correlation functions $C(0, N)$ seem to indicate that the corresponding non-linear ODEs are drastically more complicated even if N. Witte showed the existence of Garnier systems for these row correlation functions 46.
    $\dagger$ "Using a term like nonlinear science is like referring to the bulk of zoology as the study of nonelephant animals." Stanislaw Ulam (but the citation could be first a citation of Emile Borel ...).

