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Noetherian mappings

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Abstract

We have introduced a simple family of birational transformations in the complex projective space CP_n that are generated by the product of the Hadamard inverse and an (involutive) collineation. We have been able to find the integrable subcases of the model and also interesting cases of transcendental integrability. Beyond these integrable subcases, we have been able to describe the degree growth-complexity of the iteration calculations of these birational mappings. These degree growth-complexities appear to be algebraic numbers. We also obtained some simple conjectures for the growth-complexity degrees of these birational transformations in CP_n for arbitrary values of n. For the two-dimensional mappings, an equality between the (degree) growth-complexity and the topological entropy was found and we have given some conjectured closed expressions for the dynamical zeta functions.

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1. Introduction

The theory of discrete dynamical systems has developed extensively during this last decade. However few results are available in the literature for rational mappings in two dimensions [1], and very few are available for mappings in more than two dimensions. Here we describe a family of birational mappings in the *n*-dimensional complex projective space CP_n , that are built by the composition of Hadamard inverses and collineations.

To motivate this analysis, let us recall some results. Previous papers [2–10] have analyzed *birational* representations of infinite discrete symmetry groups generated by *involutions*, which have their origin in the theory of exactly solvable models in lattice statistical mechanics. These involutive birational mappings, which generate these discrete

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symmetries of the parameter space of the models, are associated with the so-called *inversion relations* [11] on vertex models, or spin models.

For vertex models, these involutions correspond, respectively, to two kinds of transformations on $q \times q$ matrices: the inversion of the $q \times q$ matrix and a permutation¹ of the entries of the matrix (corresponding to the parameter space of the model).

For *edge* spin models [17,18], the two involutions one composes correspond, for a $q \times q$ matrix M with complex *homogeneous* entries m_{ij} , to the matrix inversion $I : M \to M^{-1}$, together with the transformation J that inverts each entry of the matrix, i.e., $J : m_{ij} \to 1/m_{ij}$ (Hadamard inverse).

For instance, for a 6×6 matrix of the form

1	r	у	Z	у	Z	Z
1 2	3	x	у	Z	у	Z
2	V	z	х	z	z	у
2	V	z	Z	х	z	у
2	3	y	z	у	х	z
1	3	z	у	z	у	x)

which is a stable pattern by I (and J of course) and corresponds to a six-state chiral Potts model in lattice statistical mechanics [17]. The explicit formula for the inversion I is given explicitly, in terms of the inhomogeneous variables u = y/x and v = z/x, by:

$$I: (u, v) \to \left(\frac{-u^2 - u + 2v^2}{1 + u + 2v - u^2 - 2uv - v^2}, \frac{u^2 + vu - v^2 - v}{1 + u + 2v - u^2 - 2uv - v^2}\right).$$
(2)

The transformation J reads: $(u, v) \rightarrow (1/u, 1/v)$.

One finds that I and J and, thus, the birational transformation $K = I \cdot J$ composition of the two previous involutions, preserves the algebraic invariant

$$\Delta(u,v) = \frac{(2v^2 + 2vu - u^2 - 2u^3 - 2vu^2 + v^2u)(u - v^2)^2}{(v+u)^4(1-u)(1-v)^2},$$
(3)

which yields a *foliation* of CP_2 (the two-dimensional projective space associated with u and v) in elliptic curves, where each curve has an infinite set of birational automorphisms [11]. Introducing an (infinite order) collineation C

$$C: \quad (u,v) \to \left(\frac{1-u}{1+2u+3v}, \frac{1-v}{1+2u+3v}\right), \tag{4}$$

one can immediately verify that C intertwines I and J since one has the following relations:

$$I = C^{-1} \cdot J \cdot C. \tag{5}$$

Alternatively, the infinite-order birational transformation $K = I \cdot J$ can be written as the product $K = C^{-1} \cdot J \cdot C \cdot J$, which is reminiscent of *Noether's theorem* [19–22,72] concerning the factorization of Cremona transformations into products of quadratic [73] transformations abd collineations. (The transformation $J : (u, v) \rightarrow (1/u, 1/v)$ is the archetype of quadratic² transformations in CP₂.)

For many (non-chiral) spin-edge models (see for instance [18]), the collineation C that intertwines the matrix inversion I and the Hadamard inversion (entries inversion) J is an involution³ and, thus, the iteration of the birational

¹ Most of the time, the permutations considered in [12–16] are involutive.

² The transformation J in CP₂ is called *quadratic* because, written with homogeneous variables, it reads: $(u, v, t) \rightarrow (vt, ut, uv)$.

³ Related to the Kramers–Wannier duality, that is, to a Z_N -Fourier transform [18,23,24]. In this spin-edge models duality framework, the collineation is an involution (for non-chiral models) most of the time, and sometimes is even a transformation of order 4 [25].

transformation $K = I \cdot J = (C \cdot J)^2$ reduces to the iteration of the (generically infinite-order) birational transformation $C \cdot J$.

In the following, we will consider a family of examples of *n*-dimensional birational transformations that correspond to the product of a simple (involutive) collineation with the transformation $J: m_{ii} \rightarrow 1/m_{ii}$ and that generalize the quadratic transformation of CP_2 to CP_n . We will analyze the complexity of the mapping iteration by studying the degrees of the successive birational expressions that correspond to the iteration of our mappings. Here we will use a method that was introduced in previous papers [26-28] and is based on the examination of successive birational expressions corresponding to the iteration of some given birational mappings. When one considers the degree d(N) of the numerators (or denominators) of the corresponding successive rational expressions for the Nth iterate, the growth of this degree is (generically) exponential with N: $d(N) \sim \lambda^N$. The constant λ has been called the growth-complexity [29]. For CP₂, it is closely related to the Arnold complexity [30,31]. Let us also recall that two universal (or "topological") measures of the complexities were found to identify on specific two-dimensional examples [27–29], namely, the previous growth-complexity λ , or the (asymptotic of the) Arnold complexity [26,27,30,31], and the (exponential of the) topological entropy [26–28,32]. The topological entropy, $\ln(h)$, is associated with the exponential growth h^N of the number of fixed points (real or complex) of the Nth iterate of the mapping [27,28,33]. These papers show that the growth-complexity⁴ λ is an *algebraic* integer (i.e., a solution of a polynomial expression with integer coefficients). This is related to the fact that the generating functions of the degrees of the successive birational expressions that correspond to the iteration of these birational mappings are quite simple rational expressions with integer coefficients. We will show that similar results also hold for the *n*-dimensional mappings analyzed here.

1.1. Towards higher-dimensional generalizations: the Diller–Favre method and Noether's theorem

In the framework of birational transformations of CP₂, Diller and Favre [34] have recently introduced a cohomological approach⁵ that can reproduce, in practice, all the values of the parameters of the mapping, where the degree growth-complexity λ is diminished. This *cohomology of curves* analysis reduces to the consideration of the spectrum of a finite dimensional matrix, which explains why the previous degree growth-complexity λ is an algebraic integer. The results and theorems of Diller and Favre [34] resemble the singularity approaches performed by various authors for specific examples [35–38], the idea being to encode the growth-complexity in the analysis of a "finite object", or a "skeleton", namely, the graph of singularities, or the cohomology $H^{(1,1)}(X)$.

Unfortunately, this cohomological approach becomes extremely difficult to generalize in CP_n with n > 2. Any cohomological consideration of birational transformations in CP_n is drastically more complicated. This can be understood, heuristically, by recalling the Noether's transformation theorem⁶ of decomposition of birational transformations in CP_2 , that is, Cremona transformations (see Appendix A). The method of factorization of birational maps dates back to Noether and Fano. The problem of factoring birational maps, whose origin was Noether's theorem [21,22] on the decomposition⁷ of Cremona transformations of a plane⁸ (see also pp. 497–498 in [42]) in a product of quadratic transformations, is simple as compared to the three-dimensional case [43].

⁴ Which identifies with h for the specific mapping in [27].

⁵ Basically the study of the cohomology of curves, that is, $H^{(1,1)}(X)$, where X is a bimeromorphic map of a Kähler surface.

⁶ Noether's transformation theorem: any irreducible curve may be carried, by a factorable Cremona transformation [39], into one with none but ordinary singular points [40,41].

⁷ Note that this theorem is not an effective theorem: it says that this decomposition exists but does not give an algorithm to actually obtain this decomposition.

⁸ For plane transformations this is a result of Rosanes, Clifford, Noether, and later, but more rigorously, Castelnuovo.

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Cremona transformations of higher-dimensional projective spaces also exist, but they no longer share with the plane transformations the property of being generated by those of order 2 (the so-called quadratic transformations) [43,44]. The richer behavior of higher-dimensional Cremona transformations is connected with the greater variety of singularities that a surface can have [45–48], as Noether and Cremona remarked. In this direction, Sarkisov announced a three-dimensional generalization of the Castelnuovo–Noether theorem, the so-called Sarkisov program [49,50]. These works show that one can decompose birational maps with four types of elementary links (see Appendix A).

Naively, one can imagine that the subset of birational maps such that the Noether's decomposition still holds (i.e., birational maps that are products of collineations and generalizations to CP_n of the quadratic transformation of CP_2 , namely $(x_1, \ldots, x_n) \rightarrow (1/x_1, \ldots, 1/x_n)$) is a singled-out subset of mappings that should have simpler sets of singularities as compared to the most general birational mappings in CP_n . Hopefully, one can generalize the cohomological approach of Diller and Favre for this subset of birational maps.

In the following, we will introduce a simple family of birational transformations in CP_n (n = 2, 3, 4, ...) generated by the simple products of the Hadamard inverse and (involutive) collineations. Remarkable results for the growth-complexity, and the topological entropy, will be obtained for these "Noetherian" birational transformations. Furthermore, we will also give a list of the integrability subcases of these mappings.

2. A CP₂ birational transformation associated with an involutive collineation

Let us construct a mapping as product of two involutions, C and J, acting on CP₂. We consider the standard quadratic involution J (or Hadamard inverse) defined as follows on the three homogeneous variables (t, x, y) associated with CP₂:

$$J:(t,x,y) \to (xy,ty,tx). \tag{6}$$

We also introduce the following 3×3 matrix, acting on the three homogeneous variables (t, x, y):

$$C = \begin{bmatrix} a - 1 & b & c \\ a & b - 1 & c \\ a & b & c - 1 \end{bmatrix}$$
(7)

and the associated collineation which reads, in terms of the two inhomogeneous variables u = x/t and v = y/t

$$(u, v) \to (u', v') = \left(\frac{a + (b - 1)u + cv}{(a - 1) + bu + cv}, \frac{a + bu + (c - 1)v}{(a - 1) + bu + cv}\right).$$
(8)

In the following, since there is no possible ambiguity, we will use the same notation, C, to denote a matrix like (7) or the associated collineation (8).

If one sets the condition c = 2 - a - b, the matrix *C* becomes a "stochastic-like" matrix⁹ (the sum of the entries in each row is equal to 1) and an involutive matrix (C^2 is the identity 3×3 matrix I_d). Furthermore, its determinant is equal to +1. Under these three conditions ($C^2 = I_d$, det(C) = +1 and the "stochasticity" condition, i.e., the sum of the entries in each row equals 1), it appears that the unique non-trivial 3×3 solution, which is non-straightforwardly reducible to a 2×2 matrix¹⁰ compatible with these conditions, is given by the matrix (7).

⁹ The entries of stochastic matrices, associated with Markov chains, are probabilities and therefore are non-negative. We do not require such an assumption here and in the following.

¹⁰ Thus yielding "separable" mappings such that, for instance, the *u*-component of the two-dimensional mapping is a function of *u* only: $K(u, v) = (K_u(u), K_v(u, v)).$



Fig. 1. Phase portrait for a = 0.498 and b = c = 0.751.

(When one sets det(C) = -1 instead of det(C) = +1, one obtains another interesting family of mappings, which will be addressed elsewhere.)

The birational mapping $K = C \cdot J$, corresponding to the product of these two involutions, reads as follows, in terms of the two inhomogeneous variables u = x/t and v = y/t

$$K: (u, v) \to (u', v') = \left(\frac{auv + (b-1)v + cu}{(a-1)uv + vb + cu}, \frac{auv + bv + (c-1)u}{(a-1)uv + bv + cu}\right),\tag{9}$$

where one sets c = 2 - a - b. Note, however, that many of the results obtained in the following are independent of this condition as will be seen below. Generically, the birational mapping (9) is *not* an integrable mapping, as can be seen, for arbitrary values of *a* and *b*, in Fig. 1.

The successive iterates $K^N(u, v)$ of the birational mapping (9) are birational expressions. Their numerators and denominators are polynomial expressions in u and v whose degrees grow exponentially with the number of iterations when the mapping is not an integrable [14,51]. One can introduce various generating functions of these successive degrees of the numerators or denominators [14,51]. The growth of these degrees, also called *degree growth-complexity*, gives a "magnitude" of the topological complexity of the mapping. The degree growth-complexity will be analyzed in Section 4.

2.1. Parameter symmetries of (9)

Due to the symmetry induced by the group of permutations of the homogeneous variables (t, x, y), one deduces the following equivalence between mappings with different parameters (a, b, c)

$$\begin{aligned} (u' = v, v' = u) &\to (a' = a, b' = c, c' = b), \\ \left(u' = \frac{1}{u}, v' = \frac{v}{u}\right) &\to (a' = b, b' = a, c' = c), \\ \left(u' = \frac{1}{v}, v' = \frac{u}{v}\right) &\to (a' = c, b' = a, c' = b), \\ \left(u' = \frac{1}{v}, v' = \frac{u}{v}\right) &\to (a' = c, b' = a, c' = b), \\ \left(u' = \frac{u}{v}, v' = \frac{1}{v}\right) &\to (a' = c, b' = a, c' = b), \end{aligned}$$
 (10)

For instance, if the mapping has a given growth-complexity for the parameters (a, b, c), it will have the same growth-complexity for the parameters $(a, c, b), \ldots, (c, b, a)$.

2.2. Mapping (9) as a measure-preserving mapping

Let us note that u - 1, v - 1 and u - v are covariant under transformation (9) and, thus, the three lines u - 1 = 0, v - 1 = 0, and u - v = 0 are globally invariant under transformation (9) even if condition c = 2 - a - b is not verified. Restricted to these (globally) invariant lines, transformation (9) reduces to a linear fractional transformation and a translation for c = 2 - a - b (see below). The singled-out role played by these three lines is clear in Fig. 1. If one considers the product $\rho(u, v) = (u - 1)(v - 1)(u - v)$, a straightforward calculation shows that Jac(u, v), the Jacobian of (9), is actually equal to

$$Jac(u, v) = (a + b + c - 1)\frac{\rho(u', v')}{\rho(u, v)} = \det(C)\frac{\rho(u', v')}{\rho(u, v)},$$
(11)

where (u', v') is the image of (u, v) under the birational transformation (9). The condition c = 2 - a - b is sufficient to have a measure-preserving transformation [32,52]: the measure that is preserved by transformation (9) is $d\mu = du dv/\rho(u, v)$. This measure is shown clearly in Figs. 1 and 2, where there is a spray of points concentrated near the three lines u - 1 = 0, v - 1 = 0, and u - v = 0. Because the mapping is two-dimensional, the relation (11) means that the birational mapping can be transformed (up to a continuous change of variable) into an area-preserving map [32,52].

When the determinant det(*C*) is equal to +1, the restriction of mapping (9) to the three globally invariant lines u = 1, v = 1, and u = v reduces to a translation. On the line u = 1, the mapping can be written as the simple translation $v_r \rightarrow v_r + a + b - 1$ where v is replaced by $v_r = 1/(v - 1)$. On the line v = 1, the mapping becomes $u_r \rightarrow u_r + 1 - b$, where u is replaced by $u_r = 1/(u - 1)$. On the line u = v, the mapping becomes $u_r \rightarrow u_r + a - 1$, where u = v is replaced by $u_r = v_r = 1/(u - 1)$.

It may happen that for some particular values of the parameters, another measure, $d\mu_2 = du dv/\rho_2(u, v)$, is also preserved

$$\operatorname{Jac}(u, v) = \frac{\rho(u', v')}{\rho(u, v)} = \frac{\rho_2(u', v')}{\rho_2(u, v)}.$$
(12)

It is then clear that the mapping is integrable (see Section 6). For invariance, the invariant of the transformation is the ratio ρ/ρ_2 .



Fig. 2. Zoom of a phase portrait for a = 0.498 and b = c = 0.751.

2.3. A systematic (u, v)-symmetric analysis

One can try to find, systematically, all the collineations *C* such that the Jacobian of transformation $K = C \cdot J$ satisfies (11). Instead, let us consider an easier problem, namely, finding the collineations *C* such that the algebraic covariant expressions $\rho(u, v)$ are rational expressions that are *also* covariant by the Hadamard inverse J(u, v) = (1/u, 1/v). To perform an exhaustive classification of these collineations, let us restrict ourselves even further, by considering all the collineations *C* that yield a given covariant $\rho(u, v)$; for instance, the (u, v)-symmetric¹¹ covariant expression $\rho(u, v)$ of the mapping (9), namely, $\rho(u, v) = (u - 1)(v - 1)(u - v)$. In this case, one obtains a set of six 3×3 matrices which are given in Appendix B: thet are C_A to C_E and the matrix (7). These matrices reduce, up to equivalences, to only three different collineations associated with 3×3 matrices given in Appendix B, namely the matrices (7), C_A , and C_D . One easily verifies that these matrices are "stochastic-like" (the vector (1, 1, 1) is an eigenvector). When one requires that the determinant be +1, one finds that their characteristic polynomials read, respectively, $(t - 1)(t + 1)^2$ for C_A , C_B , C_C , and (7), and $(t - 1)(t^2 - t + 1)$ for C_D and C_E of Appendix B.

¹¹ Let us remark that $\rho(u, v) = (u - 1)(v - 1)(u - v)$ is not the only covariant of (8) which is also a *J*-covariant; the individual factors u - 1, v - 1 and u - v are also covariant (u = 1, v = 1 and u = v are invariant lines), but these factors, or even the product of only two of these factors, can hardly satisfy (11).

One easily finds that matrix (7) is an involution when its determinant is ± 1 , and that matrices C_D and C_E are matrices of order 6 when their determinant is ± 1 . In contrast, the matrices C_A , C_B , C_C and (7) are infinite-order matrices when their determinant are equal to ± 1 (the matrix (7) is also infinite order when its determinant is -1).

If one requires that the determinant, det(*C*), be equal to -1, then one obtains $Jac(u, v) = -\rho(u', v')/\rho(u, v)$; thus, the map K^2 , rather than *K*, must be measure-preserving.¹² When the determinant is equal to -1, the restriction of mapping K^2 (which is the square of (9)) to the three invariant lines u = 1, v = 1, and u = v, is no longer a translation on these three lines: each point of these three lines is a fixed point of K^2 !

3. The Diller-Favre method and criterion

In the following, out aim is to describe more quantitatively the growth-complexity in terms of the parameters a, b, c, of the mappings (9). Many methods are available [27–29,51], such as counting the degrees, counting the fixed points, studying the singularities, calculating gcd's, etc. In this section, we use the Diller–Favre method [34] to describe the singularities of the mapping. For the mapping (9), we give the equivalent of Lemmas 9.1 and 9.2 in [34].

Among the cases where the complexity is diminished, the integrable cases play a special role. The integrable cases are deduced from an integrability criterion [34] using mathematical considerations in dealing with the indeterminacy sets and exceptional sets [34] of mapping K, and the *analytical stability* [34] of the mapping. In that way, one obtains all the integrable cases, together with the cases of lower degree growth-complexity. Let us apply the method to our particular mapping.

The Jacobians of transformations *K* and K^{-1} are given by:

$$J(K) = (a + b + c - 1) \frac{uv}{((a - 1)uv + cu + bv)^3},$$

$$J(K^{-1}) = (a + b + c - 1)^2 \frac{(a - 1) + bu + cv}{(a + (b - 1)u + cv)^2(a + bu + (c - 1)v)^2}.$$

Here we assume that the condition c = 2 - a - b is satisfied (meaning that the first factor a + b + c - 1, in the previous expressions of the Jacobians, is equal to 1). The Jacobian J(K) vanishes on u = 0 and on v = 0, and becomes infinite when v = (a + b - 2)u/((a - 1)u + b) = -cu/((a - 1)u + b). The Jacobian of K^{-1} , namely $J(K^{-1})$, vanishes on v = -((a - 1) + bu)/c and becomes infinite when v = (a + bu)/(a + b - 1) = (a + bu)/(1 - c) or v = (a + (b - 1)u)/(a + b - 2) = -(a + (b - 1)u)/c.

Using the same terminology as in [34], one can show that the exceptional locus¹³ of K is given by:

$$\mathcal{E}(K) = \left\{ (u=0); \, (v=0); \, \left(v = \frac{-cu}{(a-1)u+b}\right) \right\}$$

and the indeterminacy locus [34] of *K* is given by:

$$\mathcal{I}(K) = \left\{ (0,0); \left(\frac{b}{b-1}, 1\right); \left(1, \frac{c}{c-1}\right) \right\}$$

Actually, for (u, v) = (0, 0), the *u* and *v* components of *K* are both of the form 0/0, for (u, v) = (b/(b-1), 1) the *v* component of *K* is of the form 0/0 and for (u, v) = (1, c/(c-1)), the *u* component of *K* is of the form 0/0.

¹² More generally, for $K^N(u, v) = (u', v')$, one obtains $Jac(K^N)(u, v) = (a + b + c - 1)^N \rho(u', v') / \rho(u, v)$ which yields measure-preserving maps but with complex values of the parameters: $c = 1 - a - b + \omega$ with $\omega^N = 1$.

¹³ Corresponding to J(K) = 0 or $J(K) = \infty$.

Similarly, for K^{-1} , the exceptional locus and the indeterminacy locus read, respectively, as

$$\begin{split} \mathcal{E}(K^{-1}) &= \left\{ \left(v = -\frac{(a-1)+bu}{c} \right); \left(v = \frac{a+bu}{1-c} \right); \left(v = \frac{a+(b-1)u}{c} \right) \right\}, \\ \mathcal{I}(K^{-1}) &= \left\{ (\infty,\infty); \left(\frac{b-1}{b}, 1 \right); \left(1, \frac{c-1}{c} \right) \right\}. \end{split}$$

To check whether *K* is analytically stable [34], one must compute the orbit of the exceptional set $\mathcal{E}(K)$. This can be easily done using the fact that (u - 1)(v - 1)(u - v) = 0 is invariant by *K*.

It is easy to see that the successive images of u = 0 by K give

$$(0,v) \to \left(\frac{b-1}{b}, 1\right) \to \left(2\frac{b-1}{2b-1}, 1\right) \to \dots \to \left(\frac{n(b-1)}{nb-(n-1)}, 1\right),\tag{13}$$

that the successive images of v = 0 by K give

$$(u,0) \rightarrow \left(1, \frac{a+b-1}{a+b-2}\right) \rightarrow \left(1, 2\frac{a+b-1}{2(a+b)-3}\right)$$
$$\rightarrow \cdots \rightarrow \left(1, \frac{n(a+b-1)}{n(a+b)-(n+1)}\right) = \left(1, \frac{n(c-1)}{nc-(n-1)}\right)$$

and that the successive images by K of v = (a + b - 2)u/((a - 1)u + b) = -cu/((a - 1)u + b) give

$$\left(u, \frac{-cu}{(a-1)u+b}\right) \to (\infty, \infty) \to \left(\frac{a}{a-1}, \frac{a}{a-1}\right) \to \left(\frac{1}{2}\frac{2a-1}{a-1}, \frac{1}{2}\frac{2a-1}{a-1}\right)$$

$$\to \dots \to \left(\frac{(n-1)a-(n-2)}{(n-1)(a-1)}, \frac{(n-1)a-(n-2)}{(n-1)(a-1)}\right).$$
(14)

On the other hand, the successive images of v = -((a - 1) + bu)/c by K^{-1} give

$$\begin{pmatrix} u, -\frac{(a-1)+bu}{c} \end{pmatrix} \to (0,0) \to \left(\frac{a-1}{a}, \frac{a-1}{a}\right) \to \left(\frac{2(a-1)}{2a-1}, \frac{2(a-1)}{2a-1}\right)$$
$$\to \dots \to \left(\frac{n(a-1)}{na-(n-1)}, \frac{n(a-1)}{na-(n-1)}\right),$$

the successive images of v = (a + bu)/(a + b - 1) = (a + bu)/(1 - c) by K^{-1} give

$$\begin{pmatrix} u, \frac{ub+a}{-1+a+b} \end{pmatrix} \to \left(-\frac{1-b+ub}{ua-a-u}, \infty \right) \to \left(1, \frac{c}{c-1} \right) \to \left(1, \frac{1}{2} \frac{2c-1}{c-1} \right)$$
$$\to \dots \to \left(1, \frac{1}{n} \frac{nc-(n-1)}{c-1} \right)$$

and the successive images of v = (a + (b - 1)u)/(a + b - 2) = -(a + (b - 1)u)/c by K^{-1} give

$$\begin{pmatrix} u, -\frac{a+(b-1)u}{c} \end{pmatrix} \to \left(\infty, -\frac{c(u-1)}{(u-1)a-u} \right) \to \left(\frac{b}{b-1}, 1\right) \to \left(\frac{1}{2}\frac{2b-1}{b-1}, 1\right)$$
$$\to \dots \to \left(\frac{1}{n}\frac{nb-(n-1)}{b-1}, 1\right).$$

For $n \ge 2$, all these *n*th iterates (by K or K^{-1}) belong to one of the three K-invariant lines u = 1, v = 1, or u = v.

The Diller–Favre method [34] focus on the analytical stability of the birational mappings. The Diller and Favre statement is that the mapping *K* is analytically stable if and only if $K^n(\mathcal{E}(K)) \notin \mathcal{I}(K)$ (resp. $K^{(-n)}(\mathcal{E}(K^{-1})) \notin \mathcal{I}(K^{-1})$) for all $n \ge 1$. The inspection of these constraints singles out the values of the parameters *a*, *b* and *c*, of the form (N-1)/N where *N* is a positive integer. For instance, recalling (13) and (14), an image of $\mathcal{E}(K)$ by *K* can become one of the three indeterminacy points of $\mathcal{I}(K)$:

$$\begin{pmatrix} \frac{n(b-1)}{nb-(n-1)}, 1 \end{pmatrix} = \begin{pmatrix} \frac{b}{b-1}, 1 \end{pmatrix} \Rightarrow b = \frac{n}{n+1}, \\ \begin{pmatrix} \frac{(n-1)a-(n-2)}{(n-1)(a-1)}, \frac{(n-1)a-(n-2)}{(n-1)(a-1)} \end{pmatrix} = (0,0) \Rightarrow a = \frac{n-2}{n-1}$$

and similarly with K^{-1} .

In the following, we will say that the parameter *a*, *b*, or *c* is *generic* if it is *not* of the form (N - 1)/N, where *N* is a positive integer. One immediately deduces from the condition $K^n(\mathcal{E}(K)) \notin \mathcal{I}(K)$ that the mapping¹⁴ *K* is analytically stable if the parameters do not belong to three sets *S*₁, *S*₂ or *S*₃ given by:

$$S_{1}: a = \frac{M-1}{M}, b, c \text{ generic}, a+b+c=2, S_{2}: b = \frac{N-1}{N}, a, c \text{ generic}, a+b+c=2, S_{3}: c = \frac{P-1}{P}, a, b \text{ generic}, a+b+c=2,$$

where M, N, and P are *positive* integers. These three sets actually correspond to three independent cases of *lower* complexity. One has a reduction of complexity for S_1 , S_2 or S_3 . For $S_1 \cap S_2$, $S_2 \cap S_3$ and $S_1 \cap S_3$ one gets further reductions of complexities. In the next section, we will see this reduction of complexity more quantitatively, by calculating, for every case, the corresponding degree growth-complexity (Arnold complexity) [30,31].

Integrability takes place for $S_1 \cap S_2 \cap S_3$, which corresponds to a finite number of cases. This means that one has to find three positive integers M, N, P such that

$$2 - \frac{M-1}{M} - \frac{N-1}{N} - \frac{P-1}{P} = 0.$$
(15)

There is a finite set of solutions of (15), which are $(M, N, P) = (3, 3, 3), (2, 4, 4), (2, 3, 6), (\infty, 2, 2), (\infty, \infty, 1),$ and, also, all the equivalent cases $(4, 2, 4), (4, 4, 2), (2, 6, 3), (3, 6, 2), (3, 2, 6), (6, 2, 3), (6, 3, 2), (2, \infty, 2),$ $(2, 2, \infty), (\infty, 1, \infty), (1, \infty, \infty)$; see Eqs. (10). Cases like $(1, \infty, \infty)$ give trivial mappings and will not be considered. The cases such as $(2, 2, \infty)$ correspond to reductions to CP₁. For instance $(2, 2, \infty)$, i.e., a = b = 1/2, c = 1, corresponds to

$$K: (u, v) \to (u', v') = \left(-\frac{uv - v + 2u}{uv - v - 2u}, -\frac{(u+1)v}{uv - v - 2u}\right).$$
(16)

Introducing the variable

$$w = \frac{(u+1)v + 2(u+v^2)}{2(u+v^2)},\tag{17}$$

the transformation (16) reduces to a linear fractional transformation

$$K_w: (v, w) \to (v', w') = \left(\frac{(v+1)(w-1)}{(1-w)v+w}, w\right).$$
(18)

¹⁴ Let us remark that the same sets S_i can be obtained when applying the method to mapping K^{-1} instead of K. This is not true for a general mapping.

Note that when K^2 , but not K, is a measure-preserving map (det(C) = -1, i.e., c = -a-b), the singularity analysis is quite different and will be considered elsewhere.

Remark. A more "pedestrian" approach to find these singled-out values (N - 1)/N amounts to considering finite-order conditions for *K* and seeking curves (not points) that are solutions of these conditions (a phenomenon that occurs on the integrable mappings). Let us denote by $(K^N)_u(u, v)$ and $(K^N)_v(u, v)$ the *u* and *v* components of K^N , respectively, and consider the two polynomial conditions corresponding to the numerators of the two equations $(K^N)_u(u, v) - u = 0$ and $(K^N)_v(u, v) - v = 0$. The resultant (in *u* for instance) of these polynomial conditions factorizes into polynomial expressions in *v* (corresponding to finite-order points) and also into expressions that do not depend on *v*, like for instance, 2(a + b) - 3 for K^4 , 3(a + b) - 4 for K^5 , etc., which mean precisely $1 - 2c, 2 - 3c, \ldots, (N - 1) - Nc$.

Another "pedestrian" approach to find these singled-out values (N - 1)/N, which is related not to integrability but, rather, to a straightforward analysis of the degree growth-complexity, amounts to seeking a common polynomial factor in the numerator and the denominator of $(K^N)_u(u, v)$ (resp. the numerator and denominator of $(K^N)_v$) and calculating the resultant (in *u* for instance) of the numerator and the denominator of $(K^N)_u(u, v)$. For K^3 (N = 3) this resultant factorizes into polynomial expressions in *v* and into expressions that do not depend on *v*, like, for instance for K_u^3 : (a + b) - 2, 2(a + b) - 3, and 3(a + b) - 4. More details on these two approaches can be found in [16].

4. Degree growth-complexity of the mapping (9)

It has been recalled, in the introduction, that the (topological) complexity of a birational mapping can be evaluated by considering the degrees $d(N) \sim \lambda^N$ of the numerators (or denominators) of the corresponding successive rational expressions for the Nth iterate of the mapping [26-28]. In this respect, the introduction of the generating functions of these successive degrees d(N) has been seen to be a powerful tool to encode this complexity, to evaluate or conjecture closed algebraic formula for this degree growth-complexity λ [29], which is closely related, in two dimensions, to the Arnold complexity [30,31]. The introduction of these generating functions is motivated by the fact that they have been found to be rational expressions for all the birational mappings (and even rational) mappings we have studied [51,53], yielding the degree growth-complexity λ to be simple algebraic integers. Let us recall, again, that the previous growth-complexity [29] λ (or the Arnold complexity [26,27,30,31]) and the (exponential of the) topological entropy [26–28,32], associated with the exponential growth h^N of the number of fixed points (real or complex) of the Nth iterate of the mapping, were found to identify for specific two-dimensional birational examples [27,28]. Similarly, the evaluation of h requires the introduction of an important generating function, namely the dynamical zeta function (see below). The identification between these two topological quantities for evaluating the complexity, the degree growth-complexity λ (Arnold complexity in two dimensions) and the topological entropy is not totally understood. These $\lambda = h$ identifications will be considered in the next section for mapping (9).

For mapping (9), the degree growth-complexity can be calculated either from the iteration of mapping (9), or from a recursion (see (C.3) in Appendix C). The same singularity in the degree generating functions must occur in both cases.

Let us denote by K_u and K_v the two components of the iterate of (u, v) by K: $K(u, v) = (K_u(u, v), K_v(u, v))$. Expressions K_u and K_v are rational functions of u and v given by (9). The generating functions $G_u(x)$ (resp. $G_v(x)$) of the successive degrees of u (resp. v) in the numerator of $K_u^N(u, v)$ (which is equivalent to the iteration of the line v = constant (resp. u = constant)) read, respectively, • For the generic case (a, b, c = 2 - a - b generic):

$$G_u(x) = G_v(x) = \frac{x}{1 - 2x}.$$
(19)

• When a = (M - 1)/M (M a positive integer), b, and c = 2 - a - b generic:

$$G_u(x) = G_v(x) = \frac{x}{1 - 2x + x^{M+1}}.$$
(20)

For various values of the positive integer M, the value of λ , which characterizes the growth of the successive degrees ($\simeq \lambda^n$), belongs to the interval $(1 + \sqrt{5})/2 \le \lambda < 2$, that is, 1.618034 $\le \lambda < 2$.

• When a = (M-1)/M and b = (N-1)/N (M and N positive integers) and c = 2-a-b generic, the generating functions read as

$$G_u(x) = \frac{x - x^{N+1}}{1 - 2x + x^{M+1} + x^{N+1} - x^{M+N}}, \qquad G_v(x) = \frac{x - x^{N+3}}{1 - 2x + x^{M+1} + x^{N+1} - x^{M+N}}.$$
 (21)

The largest of the inverse of the zeros of polynomial $(1 - 2x + x^{M+1} + x^{N+1} - x^{M+N})$ in terms of *M* and *N* corresponds to the growth-complexity λ . For various values of the positive integer *M* and *N*, one has $1.324718 < \lambda < 2$. It appears from Eq. (21) that these |x| < 1 singularities of the other Eq. (20) are obtained as the limit $N \to \infty$ of (21).

Eqs. (19)–(21) have been checked, ¹⁵ up to order 8 of the iteration (K^8), for many values of M and N. When one uses recursion (C.3) of Appendix C to evaluate this degree growth-complexity, one recovers exactly the same singularity as in (19)–(21) (see Appendix C for more details).

5. Dynamical zeta functions

It is interesting to compare the previous results, giving the generating functions for the successive degrees of the iterates (Arnold complexity or growth-complexity), with the corresponding dynamical zeta functions to see if the singularities of these two sets of generating functions identify, thus yielding an identification between these two (topological) complexities: the Arnold complexity and the topological entropy [26–28].

Let us just briefly recall here, that, by analogy with the Riemann ζ function, Artin and Mazur [54] introduced a powerful object, the so-called *dynamical zeta function*:

$$\zeta(x) = \exp\left(\sum_{m=1}^{\infty} \# \operatorname{fix}(K^m) \frac{x^m}{m}\right),\tag{22}$$

where #fix(K^m), denotes the number¹⁶ of fixed points of order m.

Similarly to the previous section, the calculations of the dynamical zeta functions have been performed for *a*, *b* and *c* generic, also for a = (M - 1)/M and *b* and *c* generic, and finally a = (M - 1)/M and b = (N - 1)/N (*M* and *N* positive integers) with *c* generic. The results are as follows. For *a* and *b* generic, the expansion of the dynamical zeta function has been calculated, up to order 10, and is agreement with

$$\zeta(x) = \frac{1-x}{1-2x}.$$
(23)

For a = (M - 1)/M, for M a positive *odd* integer, and b and c generic, the expansion of the dynamical zeta functions have been calculated for various values of M (M = 3, 5, 7, ...), up to order 10, and are agreement with

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¹⁵ And numerically iterating various *rational* points up to order 15. See also Appendix C. The rapid convergence of these results conforts this conjecture.

¹⁶ If one of these numbers is infinite the definition breaks down.

the expansion of

$$\zeta_M(x) = \frac{1-x}{1-2x+x^{M+1}} = \frac{1}{1-x-x^2-x^3\cdots-x^M}.$$
(24)

For a = (M-1)/M and b = (N-1)/N, where both *M* and *N* are positive odd integers, the expansion of the dynamical zeta functions have been calculated for various *odd* values of *M* and *N* ((*M*, *N*) = (3, 5), (3, 7), (3, 9), (5, 7), (5, 9), (7, 9), ...) the dynamical zeta function, calculated up to order 10, are in agreement with the expansion of

$$\zeta_{M,N}(x) = \frac{1-x}{1-2x+x^{M+1}+x^{N+1}-x^{M+N}}.$$
(25)

For even values of M and N, the exact expressions for the dynamical zeta functions seem to be more involved, and difficult, to "guess" (see Appendix D with expansions of the dynamical zeta function for M = 2 and 4, with 2 - a - b, b generic).

However, all these results also indicate the *same* singularities as those of the degree generating functions (20) and (21), namely $1 - 2x + x^{M+1}$ and $1 - 2x + x^{M+1} + x^{N+1} - x^{M+N}$. This confirms the identification between the Arnold complexity and topological entropy, early seen on specific examples in [26–28]. In this paper, we will not try to give the dynamical zeta functions for higher-dimensional mappings (in CP_n, $n \ge 3$) but only the degree generating functions for these mappings, since the counting of fixed points for mappings of more than two variables yields larger, and more subtle, calculations (see Section 8.5).

6. Cases of integrability of mapping (9)

For the integrable cases, the generating functions $G_u(x)$ and $G_v(x)$ degenerate into rational expressions with root-of-unity singularities:

$$\begin{aligned} G_{u}^{(2,4,4)}(x) &= \frac{x(1+x^{2}+x^{3}-x^{4}+x^{5})}{(1-x)^{3}(1+x)(1+x^{2})}, \qquad G_{v}^{(2,4,4)}(x) = \frac{x(1+x+x^{2})(1-x+x^{2})}{(1-x)^{3}(1+x)(1+x^{2})} \\ G_{u}^{(2,3,6)}(x) &= \frac{x(1+x^{2}-x^{3}+x^{4})}{(1-x)^{3}(1+x+x^{2})}, \qquad G_{v}^{(2,3,6)}(x) = \frac{x(1+x+x^{2}+x^{3}+x^{4})}{(1-x)^{3}(1+x+x^{2})(1+x)}, \\ G_{u}^{(3,3,3)}(x) &= \frac{x(1+x^{2}-x^{3}+x^{4})}{(1-x)^{3}(1+x+x^{2})}, \qquad G_{v}^{(3,3,3)}(x) = \frac{x(1+x)(1-x+x^{2})}{(1-x)^{3}(1+x+x^{2})}, \\ G_{u}^{(2,2,\infty)}(x) &= \frac{x}{(1-x)^{2}}, \qquad G_{v}^{(2,2,\infty)}(x) = \frac{x(1+x^{2})}{(1-x)^{2}}. \end{aligned}$$

The last two simple generating functions, $G_u^{(2,2,\infty)}(x)$ and $G_v^{(2,2,\infty)}(x)$, correspond to linear fractional transformations (see (18)). Let us remark that for the integrable cases, the conjecture (21) does not yield the correct denominators. For example, one obtains the singularity $(1 - x - x^3)$ for (2,4,4) and (2,3,6), and $(1 - x - x^2)$ for (3,3,3). However, this is not a problem since the integrable cases are singled-out situations that are *not* obtained continuously from the other non-generic cases.

In these integrable cases, one can even calculate the corresponding algebraic invariants of the mapping. For instance, for M = N = 2 ($P = \infty$) the algebraic invariant of transformation (9) reads (17) or equivalently:

$$I_{2,2,\infty}(u,v) = \frac{v(u+1)}{(u-v)(v-1)}.$$
(26)

For M = N = P = 3, the algebraic invariant of transformation (9) reads:

$$I_{3,3,3}(u,v) = \frac{(1+u+v)(uv+u+v)}{(u-1)(v-1)(u-v)}.$$
(27)

For M = 2, N = 3, P = 6, it reads:

$$I_{2,3,6}(u,v) = \frac{(v+1)(v+3+2u)(v+2u)(2v+u)(3vu+2v+u)}{(v-1)^2(u-1)(u-v)^3}$$

and M = 2, N = P = 4, it reads:

$$I_{2,4,4}(u,v) = \frac{(v+2+u)(v+2vu+u)(v+u)}{(v-1)(u-1)(u-v)^2}$$

The invariants that correspond to the other equivalent cases can be obtained from the invariants given above by using the correspondences (10). As it has been remarked above, all these integrable cases correspond to the appearance of a new invariant measure for the mapping (9).

Pictorially, the various orbits associated with the algebraic curves corresponding to different values of $I_{3,3,3}$ (u, v), intersect at some *base points*. Therefore, these base points must correspond to indeterminacy values of



Fig. 3. The integrable case M = 3, N = 3, P = 3: a linear pencil of fifty algebraic curves.

 $I_{3,3,3}(u, v)$. Actually the base points are the points for which the numerator and the denominator of the algebraic invariant (27) are simultaneously equal to zero, yielding an indeterminate value of the algebraic invariant.

Fig. 3 shows a set of 50 orbits corresponding to the integrable case M = 3, N = 3, P = 3. They make clear the existence of a foliation of CP₂ in (algebraic) curves. These curves correspond to a so-called *linear pencil* of curves. On this figure, the existence of the so-called base points of the linear pencil is also quite clear. It is also clear that the base points are located on the three globally invariant lines u - 1 = 0, v - 1 = 0 and u - v = 0, which correspond to the denominator of the algebraic invariant (27).

When one considers the birational mapping (9) for the parameters a, b, and c near the previous integrable values, namely a = 0.666667 and b = 0.6666633 (c = 2 - a - b), one sees, with Fig. 4, that a *single* orbit of this non-integrable mapping gives a "spray" of points reminiscent of the integrable foliation of Fig. 3. The base points of the integrable foliation of Fig. 3 can clearly still be seen on the orbit of Fig. 4. Furthermore, one sees that this spray of points has a higher density on a line u + v + 1 = 0 and a hyperbola uv + u + v = 0, which actually correspond to the numerator of algebraic invariant (27). One recovers the fact that the base points are located on the simultaneous vanishing conditions of the numerators and denominators of the algebraic invariants (here (27)).



Fig. 4. Deformation of the integrable case M = 3, N = 3, and P = 3: a single orbit for a = 0.666667, b = 0.6666633 and c = 0.6666667.

7. Transcendental integrability of (9)

Let us consider the mapping defined by (7), for c = 0 (or equivalently a = 0 or b = 0)

$$K: (u, v) \to \left(\frac{au+1-a}{(a-1)u+(2-a)}, \frac{auv+(2-a)v-u}{v((a-1)u+(2-a))}\right).$$
(28)

In this case, a simplification of the mapping can be done by performing the following change of variables:

$$t = \frac{1}{(u-1)}, \qquad s = \frac{u-v}{(u-1)(v-1)}$$

One obtains a new mapping in these new variables (t, s), given by:

$$K_s: (t,s) \to \left(t' = a - 1 + t, s' = \frac{s(a-1+t)}{1+t}\right).$$
 (29)

One sees that the action on t is just a translation by (a - 1). The action of K_s^N can be easily expressed by:

$$K_{s}^{N}:(t,s)\to(t_{N},s_{N})=\left(N(a-1)+t,s\frac{\Gamma(((t+a-1)/(a-1))+N)\Gamma((t+1)/(a-1))}{\Gamma((t+a-1)/(a-1))\Gamma(((t+1)/(a-1))+N)}\right),$$
(30)

where Γ denotes the usual Gamma function. The following quantity I(a, t, s), defined for c = 0, given in terms of Gamma functions, is invariant by transformation (30)

$$I(a, t, s) = s \frac{\Gamma((t+1)/(a-1))}{\Gamma((t+a-1)/(a-1))},$$
(31)

thus providing an example¹⁷ of a *transcendental invariant* expressed in terms of transcendental functions. In this "transcendental integrability" case, however, the degrees of the numerators of the successive iterates of the mapping, have a polynomial growth, and the generating function of these degrees reads:

$$G_u^{\text{trans}}(x) = \frac{x}{(1-x)^2}.$$
 (32)

For the values of a of the form (M - 1)/M, where M is a positive or negative integer, one has a simplification of the Gamma functions, and the invariant (31), defined for c = 0, becomes a rational expression. For example,

$$I(5/4, t, s) = s(4t+1)(2t+1)(4t+3), \qquad I(4/3, t, s) = s(3t+1)(3t+2)$$
$$I(3/2, t, s) = s(2t+1), \qquad I(2, t, s) = s, \qquad I(0, t, s) = \frac{s}{t(t+1)},$$
$$I(1/2, t, s) = \frac{s}{t(t+1)(2t+1)}, \qquad I(2/3, t, s) = \frac{s}{t(t+1)(3t+2)(3t+1)}.$$

We remark that this "transcendental" integrability cases is not given by the Diller–Favre conditions [34] of Section 3.

8. A CP₃ birational transformation associated with an involutive collineation

To see how the previous results extend to CP_n , let us introduce a collineation associated with the 4 × 4 matrix, similarly to Section 2

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¹⁷ Other similar examples of transcendental invariants, associated with birational mappings, have also been obtained in [55].

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$$C = \begin{bmatrix} a-1 & b & c & d \\ a & b-1 & c & d \\ a & b & c-1 & d \\ a & b & c & d-1 \end{bmatrix},$$
(33)

where d = 2 - a - b - c. Similarly to (7), this 4×4 matrix is also an *involutive* matrix when d = 2 - a - b - c. Its determinant is equal to -1. Furthermore, C is also a "stochastic-like" matrix when condition d = 2 - a - b - c is imposed: the sum of the entries in a row is equal to +1.

Introducing, again, the Hadamard inverse J(u, v, w) = (1/u, 1/v, 1/w), the birational mapping $K = C \cdot J$, corresponding to the product of the two involutions C and J, reads:

$$(u, v, w) \rightarrow (u', v', w') = \left(\frac{auvw + (b-1)vw + cuw + duv}{(a-1)uvw + vwb + cuw + duv}, \frac{auvw + vwb + (c-1)uw + duv}{(a-1)uvw + bvw + cuw + (d-1)uv}, \frac{auvw + bvw + cuw + duv}{(a-1)uvw + bvw + cuw + duv}\right).$$

$$(34)$$

In contrast with the previous CP₂ birational mapping (9), the elimination of w and v does *not* yield a recursion on the successive u's (we can denote u_n , u_{n+1} , u_{n+2} and u_{n+3}) but yields a polynomial algebraic relation:

$$P(u_n, u_{n+1}, u_{n+2}, u_{n+3}) = 0$$

where P is a polynomial of degree 2 in u_n and u_{n+3} and of degree 6 in u_{n+1} and u_{n+2} with integer coefficients.

8.1. Mapping (34) as a measure-invariant mapping

As for mapping (9) in CP₂, one can show that the expressions u - 1, v - 1, w - 1, u - v, v - w and w - u are all *covariant* under transformation (34). If one considers $\rho(u, v, w) = (Cov(u, v, w))^{2/3}$ where:

$$Cov(u, v, w) = (u - 1)(v - 1)(w - 1)(u - v)(v - w)(u - w),$$
(35)

a straightforward calculation shows, when d = 2 - a - b - c, that the Jacobian of transformation (34), Jac(u, v, w), is actually equal to:

$$Jac(u, v, w) = \frac{\rho(u', v', w')}{\rho(u, v, w)},$$
(36)

where (u', v', w') is the image of (u, v, w) under the birational transformation (34). Condition (36) is a sufficient to have a measure-preserving transformation [32,52]: the measure preserved by transformation (34) is actually $du dv dw / \rho(u, v, w)$.

The three-dimensional mapping (34) is measure-preserving which means, again, that, up to a continuous change of variable (see [32]), the birational mapping (34) can be changed into a *volume*-preserving map. The fact that, up to change of variables, one is no longer area-preserving (as in Section 2.2) but volume-preserving is clear on the "texture" of orbits of the mapping as can be seen in Fig. 5, which displays 50 orbits corresponding to the iteration of (34) with parameters a = 0.7529, b = 0.75 and c = 0.4999999, where d being deduced by the relation d = 2 - a - b - c, namely d = -0.0028999. Let us recall that, in the $d \rightarrow 0$ limit, the mapping (34) degenerates into the two-dimensional mapping (9), which is, up to a change of variable, equivalent to an area-preserving map. If, instead of the previous values for a, b and c one considers values such that d becomes "smaller", the mapping (34) tends to become, up to a change of variables, equivalent to an area-preserving map. This is clear from Fig. 6 which corresponds to a phase portrait (50 orbits) typical of area-preserving maps; Fig. 7 is intermediate between the phase portraits of area-preserving maps and volume-preserving maps.



Fig. 5. Phase portrait of the volume-preserving mapping (34), up to a change of variables.

8.2. An integrable case

In the framework of such families of measure-preserving maps, the integrability cases can be seen as the occurrence of another preserved measure. Actually, when a = b = c = d = 1/2, the mapping (34) becomes integrable and one can easily see that another independent measure is preserved. In that case, the Jacobian of *K* reads:

$$Jac(u, v, w) = 16 \frac{v^2 w^2 u^2}{(uvw - vw - uw - uv)^4}.$$
(37)

Let us introduce

$$H_u = (u+1)^2 (v+w)^2$$
, $H_v = (v+1)^2 (u+w)^2$, $H_w = (w+1)^2 (u+v)^2$

These three expressions are covariant expressions for transformation (34) and are such that

$$\operatorname{Jac}(u, v, w) = \frac{H_u(u', v', w')}{H_u(u, v, w)} = \frac{H_v(u', v', w')}{H_v(u, v, w)} = \frac{H_w(u', v', w')}{H_w(u, v, w)},$$
(38)



Fig. 6. Phase portrait of mapping (34): the area-preserving limit.

where (u', v', w') is the image of (u, v, w) under the birational transformation (34). From these equalities (38), one deduces easily the following algebraic invariants of transformation (34):

$$I_u = \frac{(w+1)(u+v)}{(v+1)(u+w)}, \qquad I_v = \frac{(u+1)(v+w)}{(w+1)(u+v)}, \qquad I_w = \frac{(v+1)(u+w)}{(u+1)(v+w)},$$
(39)

the product of these three expressions being equal to +1, $I_u I_v I_w = 1$. Therefore, the orbits of the iteration of (34) are algebraic *curves* having the following equations: $I_u = \rho$ and $I_v = \mu$ where ρ and μ are two constants depending on the initial point (u, v, w) in the iteration.

If one uses the second equation $I_v = \mu$ to eliminate the variable w, one obtains a reduced integrable (birational!) mapping in u and v

$$(u, v) \to (u', v') = \left(\frac{\mu u^2 - v^2 u^2 + v^2 - \mu v^2}{v^2 u^2 - 2\mu v u^2 - 2\mu v^2 u - v^2 + 2\mu v u + \mu v^2 + \mu u^2}, \frac{\mu v^2 - v^2 u^2 - 2v^2 u - v^2 + 2u^2 v + 2v u - \mu u^2}{v^2 u^2 - 2\mu v u^2 - 2\mu v u^2 - v^2 + 2\mu v u + \mu v^2 + \mu u^2}\right)$$
(40)

with the following algebraic invariant (deduced from I_u after elimination of w):



Fig. 7. Phase portrait of mapping (34): towards area-preserving limit.

$$I_{\text{reduced}}(u, v) = \frac{(u+v)(v-1)(u+1)}{(\mu u^2 - u^2 - u + \mu vu + vu + v - \mu u - \mu v)(v+1)}$$

From this invariant, one can obtain the base points of the projection on the (u, v) plane of the orbits of (34): (u, v) = (0, 0), (-1, 1), (1, 1), (-1, 1) and (-1, -1).

Fig. 8 shows a set of 50 orbits of mapping (34), which make clear the integrability of the mapping and, again, make clear the existence of base points for the (two-dimensional) *projection* of these orbits.

8.3. Growth-complexity of mapping (34)

The following calculations of the degree generating functions have been obtained by iterating a parametric curve in t, (u(t), v(t), w(t)) (which is a line most of the time), and then counting the intersections of its iterates with another fixed line $(v = \mu)$, for example, where μ is a constant). This amounts to calculating the degree of t in the numerator of u(t), for example.

Semi-numerical results, detailed in Appendix E, show that one has, as an extension of the Diller–Favre conditions of Section 3, diminished complexities for the various values of (a, b, c, d) = (a, b, c, 2 - a - b - c) where a, b, c, or d are of the form (N - 1)/N (N a positive integer). One has the following denominators for the degree generating functions for the various values of (a, b, c, 2 - a - b - c):



Fig. 8. Projection of 50 orbits of (34): an integrable foliation for (40).

- a, b, c and d = 2 a b c all generic: (1 3x), yielding $\lambda = 3$.
- a = (N-1)/N and all the other parameters b, c and d = 2 a b c generic:
 - N = 2, i.e., a = 1/2: $(1 3x + 2x^3)$, giving a growth $\lambda \simeq 2.73205$,
 - N = 3, i.e., a = 2/3: $(1 3x + 2x^4)$, giving a growth $\lambda \simeq 2.91963$,
 - N = 4, i.e., a = 3/4: $(1 3x + 2x^5)$, giving a growth $\lambda \simeq 2.97445$.

We conjecture that the denominators of the generating functions should be, for arbitrary N: $(1 - 3x + 2x^{N+1})$. • a = b = (N - 1)/N and c, d generic:

- N = 2, i.e., a = b = 1/2: $(1 3x + 4x^3 x^4 x^5)$, giving $\lambda \simeq 2.41421$,
- N = 3, i.e., a = b = 2/3: $(1 3x + 4x^4 x^6 x^7)$, giving $\lambda \simeq 2.83118$,
- N = 4, i.e., a = b = 3/4: $(1 3x + 4x^5 x^8 x^9)$, giving $\lambda \simeq 2.94771$.
- a = (N 1)/N, b = (M 1)/M and *c*, *d* generic:
 - N = 2 and M = 3, i.e., a = 1/2, b = 2/3: $(1 3x + 2x^3 + 2x^4 x^5 x^6)$, giving $\lambda \simeq 2.62966$,
 - N = 3 and M = 4, i.e., a = 2/3, b = 3/4: $(1 3x + 2x^4 + 2x^5 x^7 x^8)$, giving $\lambda \simeq 2.89089$,
 - N = 2 and M = 4, i.e., a = 1/2, b = 3/4: $(1 3x + 2x^3 + 2x^5 x^6 x^7)$, giving $\lambda \simeq 2.69679$,
 - N = 2 and M = 5, i.e., a = 1/2, b = 4/5: $(1 3x + 2x^3 + 2x^6 x^7 x^8)$, giving $\lambda \simeq 2.71951$.

We conjecture for these two previous cases (a = (N-1)/N, b = (M-1)/M and c, d = 2 - a - b - c generic)

that the polynomial

$$P_{N,M}(x) = 1 - 3x - x^{N+M} - x^{N+M+1} + 2x^{N+1} + 2x^{M+1}$$

is the denominator of the degree generating function.

- a = b = (N 1)/N, c = (P 1)/P and d = 2 a b c generic: • a = b = 1/2, c = 2/3: $(1 - 3x + x^2 + x^3 + 2x^4 - 2x^5)$ with $\lambda \simeq 2.26953$.
- *a*, *b*, *c*, *d*, *all* of the form (N 1)/N: These are all integrable cases. Except a new solution (2, 2, 2, 2) (i.e., a = b = c = d = 1/2), all the other integrable cases correspond to having one of the parameters *a*, *b*, *c*, *d* equal to 0; the mapping thus reduces to an integrable mapping (9) in CP₂. One thus obtains the extensions of solutions already obtained for CP₂, up to the previous new genuinely CP₃ solution (2, 2, 2, 2).

For more detail on these semi-numerical calculations, see Appendix E.

Let us finally note that these results are completely in agreement with the Diller–Favre conditions [34], which have been proved only for CP_2 , even-though they also seem to apply for our class of particular mappings of CP_n constructed as products of collineations and Hadamard inverses.

8.4. Conjectures on complexities

Conjecture 1. In view of all the previous semi-numerical results, we conjecture, for CP₃, that the denominators of the generating functions (which corresponds to the polynomial associated to the complexities) should be for a = (M - 1)/M, b = (N - 1)/N, c = (P - 1)/P and d generic, as follows (M, N, P positive integers):

$$D_{N,M,P}(x) = 1 - 3x - (1+x)(x^{M+P} - x^{N+M} - x^{P+N}) + 2(x^{N+1} + x^{M+1} + x^{P+1}) + 2x^{N+M+P}.$$
 (41)

 $D_{N,M,P}(x)$ becomes for P = 1 (i.e., for c = 0, corresponding to a reduction to a two-dimensional mapping):

$$D_{N,M,1}(x) = (1 - 2x + x^{N+1} + x^{M+1} - x^{N+M})(1 - x).$$
(42)

One recovers, in this c = 0 limit, the CP₂ conjecture (see Eq. (21)).

Conjecture 2. The case a = (M - 1)/M, b = (N - 1)/N (M, N are positive integers), c and d generic, can be obtained by setting the $P = \infty$ limit in (41), namely

$$D_{N,M}(x) = 1 - 3x - x^{N+M} - x^{N+M+1} + 2x^{N+1} + 2x^{M+1}.$$
(43)

The case a = (M-1)/M, b, c, d generic can be obtained by setting the limits $N = \infty$ and $P = \infty$ in (41), namely

$$D_M(x) = 1 - 3x + 2x^{M+1}.$$
(44)

When considering all the possible values of the (positive) integers N, M, and P, one finds that the growth-complexity λ belongs for (41) to the interval 2.26953 < λ < 3, for (43) to 2.41421 < λ < 3, and for (44) to 2.73205 < λ < 3.

Let us remark that all these conjectures are valid in "generic enough" cases, that is, when the complexity is reduced from $\lambda = 3$ to various algebraic integers according as one of the three parameters *a*, *b*, *c* is of the form (M-1)/M. However, when the mapping becomes *integrable*, these conjectures are *no longer valid*. For instance, when M = N = P = 2 (which is the only new integrable case beyond reductions to the mapping (9) of CP₂), conjecture (41) is not valid. In that case, the conjectured polynomial (41) factorizes into $(1 - 2x)(1 + x)^2(1 - x)^3$, which is not compatible with the integrability of the mapping (see Section 8.2), similarly to what happened with the two-dimensional mapping (9) (see also Section 6). Another integrable case is, for instance, when N = M = P = 3,

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which corresponds to d = 2 - a - b - c = 0, and, thus, to a reduction to the mapping (9) of two variables of CP₂ (see Section 8.6 and Fig. 3). The conjectured polynomial (41) is not valid in this integrable case, since it factorizes as $(1 - 2x - 2x^2)(1 + x + x^2)^2(1 - x)^3$: a polynomial growth of the calculations gives only singularities on the unit circle.

8.5. Dynamical zeta functions for the three-dimensional mappings

Similarly to the calculations performed in Section 5, it is tempting to try to calculate the dynamical zeta function for the three-dimensional mapping (34), and *if* the dynamical zeta function is a simple enough *rational* function, compare the denominators of this rational function with the ones previously conjectured for the degree generating functions (41)–(44). When performing these calculations, one immediately faces the problem that the Weil cycle decomposition way of calculating the expansions of the dynamical zeta functions [26] becomes much more subtle,¹⁸ and involved, in more than two dimensions. Basically, one finds in three dimensions (and it is even more complicated in higher dimensions) that the set of fixed points of the mapping is "stratified" in algebraic varieties of various dimensions. Actually, for the three-dimensional mapping (34) the set of fixed points is isolated points and *also* algebraic curves of fixed points.

Before sketching a specific example (mapping (34) for N = 2, M = 2 and P = 3), let us recall the Weil cycle decomposition of dynamical zeta functions [26]. An alternative way of writing the dynamical zeta functions relies on the decomposition of the fixed points into *cycles*, which corresponds to the Weil conjectures [56]. Let us introduce N_r , the number of irreducible cycles of K^r : for instance, for N_{12} , we count the number of fixed points of K^{12} that are not fixed points of K, K^2 , K^3 , K^4 or K^6 , and divide by 12. One can write the dynamical zeta function as

$$\zeta(x) = \frac{1}{(1-x)^{N_1}} \frac{1}{(1-x^2)^{N_2}} \frac{1}{(1-x^3)^{N_3}} \cdots \frac{1}{(1-x^r)^{N_r}} \cdots$$
(45)

The combination of the N_r 's, inherited from the product (45), automatically takes into account the fact that the total number of fixed points of K^r can be obtained from fixed points of K^p , where p divides r, and from irreducible fixed points of K^r itself (see [56] for more details).

Let us now consider, for instance, K_{223} , which denotes the mapping (34) for N = 2, M = 2, and P = 3. The fixed points of K_{223} are *not* isolated points but are *all* the points of the line (u, v, w) = (1, t, t). Instead of the (complex and real) isolated points one could expect for the fixed points of K_{223}^2 , one finds a *curve* of fixed points of K_{223}^2 , namely the rational curve Γ :

$$u(t) = \frac{(t+2)(t-1)}{3t+5}, \qquad v(t) = \frac{(t+3)(1-t)}{4(t+1)}, \qquad w(t) = -2\frac{(t+3)(t+2)}{(3t+5)(t+1)}.$$
(46)

Each point of Γ is a fixed point of K_{223}^2 . One could imagine, at first sight, and in a naive cycle viewpoint, that the rational curve Γ transforms by K_{223} into another (rational) curve Γ' , this curve being also transformed into Γ by K_{223} . In fact, one finds the following action of K_{223} on Γ :

$$K_{223}: (u(t), v(t), w(t)) \to \left(\frac{1}{u(t)}, \frac{1}{v(t)}, \frac{1}{w(t)}\right).$$
(47)

In other words, on the rational curve (46), the action of K_{223} identifies with the action of the Hadamard inverse J. Furthermore, the points (1/u(t), 1/v(t), 1/w(t)) do not correspond to a new curve Γ' but actually belong to (46). The transformation K_{223} , or the Hadamard inverse J, is, in fact, represented on (46) by the involutive automorphism:

¹⁸ This can also be seen in the Diller–Favre cohomological approach [34]: the cohomology is drastically more complicated.

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$$K_{223} \text{ or } J: t \to t' = \frac{t+7}{t-1}.$$
 (48)

However, since K_{223} is the product of the involution J and of a collineation C_{223} , the rational curve (46) must be a set of fixed points of C_{223} . The fixed points of the collineation C_{223} correspond to the hyperplane 3u + 4v + 2w + 3 = 0. The rational curve (46) thus corresponds to points belonging to this hyperplane and such that Hadamard inverse *also* belongs to this hyperplane

$$3u + 4v + 2w + 3 = 0$$
 and $\frac{3}{u} + \frac{4}{v} + \frac{2}{w} + 3 = 0$

With this example, one sees that the counting of cycles of "fixed algebraic curves" has to be performed carefully: the action of a birational mapping *K* on algebraic curve Γ_1 of fixed points of K^{nm} can, for instance, generate *n* different curves Γ_{α} ($\alpha = 1, ..., n$), each of these curves being preserved by automorphisms of order *m*.

For the three-dimensional mapping (34), one sees from the previous calculations (46) that it may be interesting to define two dynamical zeta functions: a dynamical zeta function ζ_{point} associated with counting the number of isolated fixed points (45), and a dynamical zeta function ζ_{curve} associated with counting the number of "fixed algebraic curves" such as (46).

In the generic case, one obtains one fixed point of K, namely (u, v, w) = (1, 1, 1). For K^2 , one could expect, at first sight, at least six cycles of isolated points, corresponding to the six reductions of (34) into (9) for u = v, w = v, u = w, u = 1, v = 1, and w = 1, on which (34) becomes the generic mapping (9) which has one cycle for K^2 . In fact, one finds that these six cycles belong to a generalization of the rational curve (46), namely an algebraic (elliptic) curve of fixed points of K^2 , defined by the equations:

$$w = uv \frac{b + c - 2 + 2a + cv + bu}{(b + c - 2 + 2a)uv + cu + bv},$$

$$E(u, v) = b(c + av)u^{2} + b(a + cv)v + ac(1 + v^{2})u - 2((ab + bc + ca) - 2(a + b + c) + 2)uv = 0.$$
(49)

A straightforward calculation shows that *each* point of the algebraic curve (49) is a fixed point of K^2 . One easily verifies that the second equation E(u, v) = 0 is an algebraic curve of genus 1, the involution $(u, v) \rightarrow (1/u, 1/v)$, leaving E(u, v) = 0 invariant, and changing w, given by (49), into 1/w. The elliptic curve (49) is thus globally invariant by the Hadamard inverse J and, thus, by the involution $C \cdot J \cdot C$.

The study of the fixed points of K^3 or K^4 yields quite large formal calculations. With the analysis of the fixed points of *K* or K^2 our expansions of the "point" or "curve" dynamical zeta functions, ζ_{point} and ζ_{curve} , or their product, are too short to compare them with simple expressions generalizing (23) like

$$\zeta_{3d}(x) = \frac{1-x}{1-3x}$$
 or $\frac{1-2x}{1-3x}, \dots$

Preliminary calculations of dynamical zeta functions of the three-dimensional mapping (34) will be detailed elsewhere.

8.6. Reductions and symmetries of the mapping

Similarly to mapping (9), one has a set of symmetries of permutations of the parameters a, b, c, and d of mapping (34). This comes from the fact that the four lines and columns of the 4×4 matrix (33) are on the same footing. Among these permutations of the four parameters a, b, c, d, one must distinguish the permutations of the three parameters b, c, d, which are clearly, and simply, associated with the permutations of the three variables u, v, w. The permutations involving a with one of the three parameters b, c, d, correspond to slightly more complicated

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transformations on the three variables u, v, w. For instance, permuting a and b is equivalent to changing (u, v, w) into (1/u, v/u, w/u). This is the straightforward generalization of the parameter symmetries (10) for the two-dimensional mapping (9). These remarks generalize, in a straightforward manner, to CP_n .

Let us now consider, with the example of mapping (34), two different types of reductions from CP_n to CP_{n-1} , namely reductions associated with limits on the four *a*, *b*, *c*, *d* parameters and reductions associated with restricting the mapping to the vanishing conditions of the covariant $\rho(u, v, ...)$.

Let us consider the first reduction, assuming that one of four parameters a, b, c, d (which are on the same footing), for instance d, is taken equal to 0. From matrix (33) one sees, immediately, that, as far as the variables u and v are concerned, the mapping (34) is identical to mapping (9), the last variable being transformed as

$$w \to 1 + \frac{1 - w}{(a - 1) + bu + cv}.$$
 (50)

In CP_n , one will have a similar result, namely that if one of the n + 1 parameters of the $(n + 1) \times (n + 1)$ matrix defining the collineation is taken equal to zero, the mapping degenerates into a mapping in CP_{n-1} .

Let us now consider the second kind of reductions, by restricting the mapping (34) to the variety v = w. This condition is preserved by the mapping (34) (see also the covariance of (35)). The mapping (34) with the parameters a, b, c, d = 2 - a - b - c then reduces to mapping (9) with the parameters a' = a, b' = b, c' = 2 - a - b = c + d. This is a consequence of the fact that the v = w limit amounts to reducing matrix (33) to matrix (7) with the parameters (a, b, c) of (7) equal to (a, b, c + d). Of course, one has similar results for the $u \rightarrow 1$ limit (or $v \rightarrow 1$, etc.), corresponding to the other factors of the covariant (35).

9. Higher-dimensional mappings (CP₄)

These results generalize to CP_4 by considering a 5 \times 5 matrix generalizing (9) and (34):

$$C = \begin{bmatrix} a-1 & b & c & d & e \\ a & b-1 & c & d & e \\ a & b & c-1 & d & e \\ a & b & c & d-1 & e \\ a & b & c & d & e-1 \end{bmatrix}.$$
(51)

One easily verifies, again, that this (stochastic-like) matrix is involutive when e = 2 - a - b - c - d. The birational transformation deduced from the product of the associated collineation and the Hadamard inverse J(u, v, w, z) = (1/u, 1/v, 1/w, 1/z), reads K(u, v, w, z) = (u', v', w', z') where, for instance, u' reads:

$$u' = K_u(u, v, w, z) = \frac{auvwz + (b-1)vwz + cuwz + duvz + euvw}{(a-1)uvwz + bvwz + cuwz + duvz + euvw}, \dots$$

9.1. A hypervolume-preserving property

Similarly to CP₂ and CP₃, one immediately verifies that u - 1, v - 1, w - 1 and z - 1 and also u - v, v - w, w - z, and z - u, are covariant under the action of (9). The following polynomial:

$$Cov(u, v, w, z) = (u - 1)^2 (v - 1)^2 (v - 1)^2 (z - 1)^2 (u - v)^3 (v - w)^3 (w - z)^3 (z - u)^3$$
(52)

associated with the cyclic permutation symmetry $(u, v, w, z) \rightarrow (v, w, z, u)$ is covariant by (9). One sees that the Jacobian of transformation (9) is given by:

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$$\operatorname{Jac}(u, v, w, z) = (a + b + c + d + e - 1) \left(\frac{\operatorname{Cov}(u', v', w', z')}{\operatorname{Cov}(u, v, w, z)} \right)^{1/4},$$
(53)

where (u', v', w', z') denotes the image of (u, v, w, z) under the birational transformation (9). One thus deduces that, provided e = 2 - (a + b + c + d), the birational transformation (9) is hypervolume-preserving map, up to a change of variables.

9.2. New invariants for mapping (9)

The variables u, v, w, z are, however, on the same footing: one does not have only the cyclic symmetry $(u, v, w, z) \rightarrow (v, w, z, u)$ corresponding to covariant (52), but the full group S_4 of permutations of these four variables.¹⁹ Therefore, one can also introduce *all* the covariants u - w, v - z, etc., and their associated cofactors, for example,

$$K: u - w \to C_{uw}(u, v, w, z)(u - w),$$

$$C_{uw}(u, v, w, z) = \frac{vz}{(a - 1)uvwz + bvwz + cuwz + duvz + euvw},$$
(54)

and similar expressions C_{uv} , C_{vz} , etc., sharing the *same* denominator (which is also the denominator in mapping (9)). One then immediately deduces, from the simple product form of their numerators, relations such as:

$$C_{uw}(u, v, w, z)C_{zv}(u, v, w, z) = C_{zu}(u, v, w, z)C_{vw}(u, v, w, z),$$
(55)

which means that the following expression is actually an algebraic invariant of transformation (9):

$$I_4(u, v, w, z) = \frac{(z - v)(u - w)}{(z - u)(v - w)}.$$
(56)

Note that this last result is valid even if relation e = 2 - (a + b + c + d) is not verified.

As a consequence, up to a change of variables, the hypervolume-preserving property of this birational mapping of four variables reduces to a volume-preserving property (up to a change of variables).

9.3. Restrictions to invariant varieties

Restricting the mapping to the invariant variety $I_4(u, v, w, z) = \rho$, where ρ is a constant, mapping (9) now reads $k_{\rho}(u, v, w) = (u', v', w')$ with, for instance, on the first coordinate:

$$u' = \frac{\rho u(v-w)G_1(u,v,w) - v(u-w)G_2(u,v,w)}{\rho u(v-w)G_3(u,v,w) - v(u-w)G_4(u,v,w)},$$
(57)

with

 $G_{1}(u, v, w) = auvw + duv + cuw + bvw + (e - 1)vw,$ $G_{2}(u, v, w) = auvw + duv + cuw + euw - (b - 1)vw,$ $G_{3}(u, v, w) = (a - 1)uvw + duv + cuw + (b + e)vw,$ $G_{4}(u, v, w) = (a - 1)uvw + duv + (c + e)uw + bvw.$

This CP₃ restriction is also measure-preserving for e = 2 - a - b - c - d, as can be seen directly calculating the Jacobian Jac(u, v, w) of this reduced transformation (57). Introducing the covariant expression:

$$Cov(u, v, w) = (u - v)(v - w)(w - u)(u - 1)(v - 1)(w - 1),$$
(58)

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¹⁹ And even a "hidden" S_5 of permutation (just think projectively).

one finds that the Jacobian Jac(u, v, w) of the reduced transformation (57) is actually such that:

$$Jac(u, v, w) = (a + b + c + d + e - 1) \left(\frac{Cov(u', v', w')}{Cov(u, v, w)}\right)^{2/3},$$
(59)

which is in complete agreement with relation (36) obtained for CP₃.

9.4. Conjecture on the complexity degrees

Finally, for CP₄, we conjecture that the growth complexities λ of mapping (9) are related to the zeros of the following polynomial:

$$D_{CP_4}(x) = 1 - 4x + 3(x^{M+1} + x^{N+1} + x^{P+1} + x^{Q+1}) - 3x^{M+N+P+Q} - (1 + 2x)(x^{M+N} + x^{M+P} + x^{M+Q} + x^{N+P} + x^{N+Q} + x^{P+Q}) + (2 + x)(x^{M+N+P} + x^{M+P+Q} + x^{M+N+Q} + x^{N+P+Q}).$$
(60)

The validity of this conjecture has been tested numerically, up to K^6 , for many values of M, N, P, and Q. Expression (60) reduces to conjecture (41) in the Q = 1 (resp. P = 1, etc.) subcase.

10. Higher-dimensional mappings CP_n , n > 4

The previous results generalize to CP_n by considering a $(n + 1) \times (n + 1)$ matrix generalizing (9), (34) and (51), such that the entries in each column are equal and the sum of the entries in each row is normalized to 2, and then subtracting the $(n + 1) \times (n + 1)$ identity matrix.

10.1. New invariants for CP_n , n > 4

Many new invariants can be given for n > 4; however, only (n - 3) of these are algebraically independent. When taking into account these algebraically independent invariants, the mapping in CP_n always reduces to a mapping in CP_3 . Let us show this explicitly for n = 5. The expression $P_1^2 P_2^{n-1} = P_1^2 P_2^4$, where

$$P_1(u, v, w, x, y) = (u - 1)(v - 1)(w - 1)(x - 1)(y - 1),$$

$$P_2(u, v, w, x, y) = (u - v)(v - w)(w - x)(x - y)(y - u)$$

is covariant and, in a similar way as in Section 9.2, one can obtain several algebraic invariants for the mapping, as, for example

$$I_{5}^{(1)}(u, v, w, x, y) = \frac{(y - w)(x - v)}{(x - y)(v - w)}, \qquad I_{5}^{(2)}(u, v, w, x, y) = \frac{(v - w)(u - y)}{(u - w)(y - v)},$$

$$I_{5}^{(3)}(u, v, w, x, y) = \frac{(u - w)(x - v)}{(u - v)(x - w)}.$$
(61)

However, these invariants are not algebraically independent. They satisfy algebraic relations such that

$$I_5^{(1)}I_5^{(2)}I_5^{(3)} + I_5^{(1)}I_5^{(3)} - I_5^{(2)}I_5^{(3)} - I_5^{(1)} - I_5^{(3)} = 0$$
(62)

and, hence, only two of them are algebraically independent. Again these results do not require the sum in a row of the entries of the $(n + 1) \times (n + 1)$ matrix to be normalized to 1.

By composing these invariants, which are associated to elementary transpositions of two coordinates, one obtains other algebraic invariants associated, for instance, with permutations of three coordinates:

$$I_5^{\text{cycle}}(u, v, w, x, y) = \frac{(u-v)(x-w)(u-y)}{(y-v)(u-w)(u-x)}.$$
(63)

10.2. Hypervolume-preserving property for CP_n

A straightforward generalization of the results of the previous section shows that $u_1 - 1$, $u_2 - 1$, $u_3 - 1$, ..., $u_n - 1$, and also $u_1 - u_2$, $u_2 - u_3$, $u_3 - u_4$, ..., $u_{n-1} - u_n$, are covariant expressions of the birational mapping generalizing (9) and (34), and, similarly, the straight generalization of ρ (see (35), (36)), namely

$$\rho(u_1, u_2, \dots, u_n) = \left(\prod_{i=1}^n (u_i - 1)^2 (u_i - u_{i+1})^{n+1}\right)^{1/n}$$
(64)

with $u_{n+1} = u_1$, is such that the Jacobian of the birational transformation is of the form $\rho(u'_1, u'_2, \dots, u'_n) / \rho(u_1, u_2, \dots, u_n)$ where $(u'_1, u'_2, \dots, u'_n)$ denotes the image of (u_1, u_2, \dots, u_n) by the birational transformation in CP_n.

The birational transformation is the product of an involutive collineation and of the Hadamard inverse $(J(u_1, u_2, ..., u_n) = (1/u_1, 1/u_2, ..., 1/u_n))$ which is also a measure-preserving transformation associated with the measure $d\mu_J = \prod_{i=1}^n du_i/u_i$ (for *n* even).

However, the measure-preserving character of our multi-dimensional birational transformations is, in fact, the consequence of the fact that the covariant $\rho(u_1, u_2, ..., u_n)$, closely related to the preserved measure, is *also* covariant for the Hadamard inverse *J* and the collineation *C* separately. Let us calculate the cofactor of the covariant $\rho(u_1, u_2, ..., u_n)$ for mapping $K = C \cdot J$ seen as a successive product of *J* and *C*

$$J: \rho(u_1, u_2, \dots, u_n) \to \rho\left(\frac{1}{u_1}, \frac{1}{u_2}, \dots, \frac{1}{u_n}\right) = (u_1 u_2 u_3 \cdots)^N \rho(u_1, u_2, \dots, u_n),$$

$$C: \rho(u_1, u_2, \dots, u_n) \to (u_1 u_2 u_3 \cdots)^N \det(C) \rho(u'_1, u'_2, \dots, u'_n).$$
(65)

Denoting by Jac[C] and Jac[J] the Jacobian of the collineation C and transformation J, one sees that these two transformations satisfy a "pre-measure-preserving property" (11)

$$Jac[C](u_1, u_2, ..., u_n) = -det(C) \frac{\rho(C(u_1, u_2, ..., u_n))}{\rho(u_1, u_2, ..., u_n)},$$

$$Jac[J](u_1, u_2, ..., u_n) = -\frac{\rho(J(u_1, u_2, ..., u_n))}{\rho(u_1, u_2, ..., u_n)}.$$
(66)

The Jacobian of transformation $K = C \cdot J$ can easily be deduced from (66)

$$Jac[K](u_1, u_2, ..., u_n) = Jac[C](J(u_1, u_2, ..., u_n))Jac[J](u_1, u_2, ..., u_n) = det(C) \frac{\rho(C(J(u_1, u_2, ..., u_n)))}{\rho(J(u_1, u_2, ..., u_n))} \frac{\rho(J(u_1, u_2, ..., u_n))}{\rho(u_1, u_2, ..., u_n)} = det(C) \frac{\rho(K(u_1, u_2, ..., u_n))}{\rho(u_1, u_2, ..., u_n)}.$$
(67)

The transformation $K = C \cdot J$ is measure-preserving when $\det(C) = +1$. In fact, one could try to find systematically the measure-preserving mappings of the form $K = C \cdot J$, such that an algebraic covariant $\rho(u_1, u_2, ..., u_n)$ is, separately, covariant by J and C and, then, such that condition (67) is verified (see Appendix B for CP₂). Combining the results of this section with those of Section 10.1, one sees that these CP_n mappings can be reduced, up to a change of variables, to volume-preserving maps.

10.3. Conjecture on degree complexity for higher-dimensional mappings

Expressions (41), (42), (44) and (60), giving the degree growth-complexity λ for a particular set of values of the parameters, can be generalized for CP_n, as follows. If *n* of the *n* + 1 parameters of the collineation matrix *C* are of the form:

$$a_1 = \frac{N_1 - 1}{N_1}, a_2 = \frac{N_2 - 1}{N_2}, \dots, a_n = \frac{N_n - 1}{N_n},$$
(68)

where N_1, \ldots, N_n are positive integers and the last parameter a_{n+1} is deduced from

$$\sum_{i=1}^{n} \frac{N_i - 1}{N_i} + a_{n+1} = 2,$$
(69)

the polynomial generalizing (41) or (60) becomes

$$D_{CP_n}(x) = 1 - nx - \sum_{i=1}^{n} (-1)^i ((n-i)x + (i-1))S_i(N_1, N_2, \dots, N_n; x),$$
(70)

where

$$S_{1}(N_{1}, N_{2}, ..., N_{n}; x) = \sum_{i=1}^{n} x^{N_{i}},$$

$$S_{2}(N_{1}, N_{2}, ..., N_{n}; x) = \sum_{i_{1} > i_{2}} x^{N_{i_{1}} + N_{i_{2}}}, ...,$$

$$\vdots$$

$$S_{k}(N_{1}, N_{2}, ..., N_{n}; x) = \sum_{i_{1} > i_{2} > i_{3} > ... > i_{k}} x^{N_{i_{1}} + N_{i_{2}} + ... + N_{i_{k}}},$$

$$\vdots$$

$$S_{n}(N_{1}, N_{2}, ..., N_{n}; x) = x^{N_{1} + N_{2} + ... + N_{n}}.$$
(71)

 $S_k(N_1, N_2, \ldots, N_n; x)$ is an expression of x containing C_k^n monomials of x.

When not all the parameters are of the form (68), then one has to use the limits $N_i \rightarrow \infty$ for those parameters which are not of the form (68), as explained in Section 8 for CP₃.

When all the parameters are not of the form (N - 1)/N ($N \ge 0$), then $D_{CP_n}(x) = 1 - nx$, giving a maximal growth-complexity $\lambda = n$. As far as the growth-complexity λ is concerned, the minimal non-trivial λ is given for CP_n from the polynomial

$$P_n(x) = 1 - (n-2)x - (n-1)x^2 - (n-1)x^3,$$
(72)

giving $\lambda_{\min} \simeq 3.2206928$, 4.18438717, 5.157447054, for respectively CP₄, CP₅ and CP₆. For CP_n, with *n* large, this gives $\lambda_{\min} \rightarrow n - 1 + 1/n - 2/n^3 - 1/n^4 \cdots$.

Remark. Integrability. It has been seen, in Section 8.6, that when one of the parameters a, b, \ldots of the mapping is equal to zero, the *n*-dimensional mapping reduces to an (n - 1)-dimensional mapping. Any integrable case of an *n*-dimensional mapping can thus be seen as an integrability case for an (n + 1)-dimensional mapping with one of its parameters being equal to zero. In Section 8.2 a *genuine* three-dimensional integrability (not reducing to a two-dimensional mapping in some way) was found for the example a = b = c = d = 1/2. In fact, it has been seen

that integrability corresponds to the situation where all the n + 1 parameters a_1, \ldots, a_{n+1} are of the form (68), i.e., where there exist n + 1 positive integers N_i such that

$$\sum_{i=1}^{n+1} \frac{N_i - 1}{N_i} = 2.$$
(73)

Seeking such genuine *n*-dimensional integrability, one may try to find all the positive integers N_1, \ldots, N_{n+1} different from one ($N_i = 1$ means $a_i = 0$ and thus a reduction to an n - 1-dimensional mapping) that satisfy (73). Actually, one finds that there are no solutions of (73) with $N_i \neq 1$ when $n \ge 4$.

11. Conclusion

We have introduced a simple family of birational transformations in CP_n , generated by the simple products of the Hadamard inverse and of (involutive) collineations. Several results for the growth-complexity λ , and the dynamical zeta function (topological entropy), were obtained for these birational transformations. In particular, we have been able to produce some simple algebraic conjectures for the growth-complexity of our birational transformations in CP_n for arbitrary values of *n*. For the two-dimensional mappings we were also able to give some simple conjectures for the dynamical zeta function in agreement with the previous conjectures, and in agreement with an identification between Arnold complexity and topological entropy. The integrability cases of these mappings were given and it was found that some transcendental integrability, with a polynomial growth of the calculations, may occur for these mappings: we actually obtained a closed expression for this transcendental integrability.

These calculations can be generalized in many directions: for instance, one can imagine to find, systematically, all the measure-preserving "Noetherian maps" built from similar products of the Hadamard inverse and of a collineation, and analyze them in a similar way. One can imagine to relax the involutive character of the collineations or, even, the measure-preserving properties of the mappings. A much more ambitious goal amounts to trying to obtain the expansion of the dynamical zeta functions for higher-dimensional "Noetherian maps" (CP_n with $n \ge 3$), in order to deduce *rational conjectures* for these dynamical zeta functions and compare these rational expressions with the degree generating functions of these birational mappings.

Birational transformations in CP_n become a very large set of transformations [57,58]. Actually, it can be seen that one does *not* have a Noether theorem any longer. However, one still has much more involved decomposition theorems [59,60] explaining that birational transformations can be decomposed into Hadamard inversions, collineations, but, unfortunately, also other ("stretching") transformations. The set of birational transformations in CP_n is so large that, for n > 2, the cohomological approach of Diller and Favre [34] can no longer be applied.²⁰ However, in this paper, one sees that birational transformations generated from collineations and Hadamard inversions (or the matrix inverse [51]) yield exponential growth of the iteration calculations λ^N where λ are clearly simple *algebraic* integers (see (41)–(44), (60) and (70)), the conjectured results for n > 2 being *straight generalizations* of the results for n = 2. This strongly suggests that it should be possible to generalize the CP₂ cohomological approach of Diller and Favre [34] for some "well-suited" *subgroup* of the birational transformations in CP_n. In this respect, it is certainly interesting to consider the subset of birational transformations in CP_n which have such a *Noetherian decomposition*, that is birational transformations in CP_n which can be written as arbitrary products of collineations and of the Hadamard inverse J. We will call "Noetherian birational transformations" such transformations. Is it necessary, in order to perform such a cohomological approach for birational transformations in CP_n, to restrict, even further, this subset of Noetherian birational transformations? All the calculations performed in this paper were

²⁰ One cannot reduce to the study of the cohomology of curves. The cohomology becomes very involved.

greatly favored by the simplicity of our Noetherian mappings (in particular their measure-preserving properties, as far as the dynamics is concerned, as can be seen on the phase portraits). It is not clear if this remarkable property is a necessary ingredient in order to generalize the CP_2 cohomological approach of Diller and Favre, though it certainly simplifies the analysis of singularities. We think that the rationality of the degree generating functions, we obtained for simple particular Noetherian birational transformations, should be understood by some generalization of this cohomological approach. The question is: which additional constraints should be imposed in order to be able to describe the corresponding cohomology?

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Appendix A. Towards higher-dimensional generalization of Noether's theorem

Birational geometry really starts with Noether's paper [19] on Cremona transformations. A Cremona transformation of the projective plane is a slippery thing. It is not quite a map from CP₂ to CP₂; rather, it is a map from "almost all" of CP2 to "almost all" of CP2. To define such a transformation, Cremona took three curves of the same degree, say n, whose equations are $F_i(x, y, z) = 0, i = 1, 2, 3$, and mapped the point [x, y, z] to $[F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)]$. In modern terms, this is a map at the point [x, y, z] provided it is not the case that all the $F_i(x, y, z)$'s vanish there, i.e., unless the point [x, y, z] lies on all three curves: such a point is called a base point. Birational geometry remained for a long time a "sleeper" probably, as far as mathematicians are concerned, because of the "slippery" nature of the transformations at first sight (i.e., proliferation of singularities: is the iteration of birational transformation well-defined on a Zariski set?). The modern period of birational geometry really started with Manin's papers on geometry of surfaces over non-closed fields [61,62]. The breakthrough into higher dimensions was made in 1970 in the papers of Iskovskikh and Manin [63]. In Iskovskikh and Manin [63], using certain ideas of Noether and Fano, developed a new method of study of birational correspondences between algebraic varieties, which have no non-trivial differential-geometric birational invariants: the method of maximal singularities. The results, which were obtained by means of this method in the seventies, were summed up in 15 years ago in [64,65]. Since that day, considerable progress has been made in the field. It is worth noting that, although we have now new approaches and concepts [66], this method, up to this day, is the most effective tool in the birational geometry.

As far as generalizations of the Castelnuovo–Noether theorem and decompositions of Cremona transformations are concerned, let us recall the following. It is known that any birational map between CP₁-bundles over a smooth curve (which are classical examples of Mori fiber spaces) can be decomposed into elementary transformations. One of the main problems is to investigate birational maps between Mori fiber spaces. In this direction, Sarkisov announced a three-dimensional generalization of the Castelnuovo–Noether theorem, the so-called Sarkisov program [49,50]: if one considers factoring birational maps of threefolds after Sarkisov, a birational transformation between minimal models is an isomorphism in codimension one and is a composition of flips or flops [67]. Sarkisov [44,49,50] introduced a notion of elementary map between Mori fiber spaces and announced a proof that every birational transformation between threefold Mori fiber spaces is a composition of elementary links [49]. Following Sarkisov's

work, Corti gave a rigorous proof of Sarkisov's theorem [43,48,68]. These works show that one can decompose birational maps with four types of elementary links but are not concrete as in the case of CP_1 -bundles. Since then, most of the progress in the theory has been restricted to dimension three. Many applications of these methods were discovered and these led to the solution of numerous open problems in the theory of surfaces and threefolds. Some of these are reviewed in [69,70].

Appendix B. Collineations yielding measure-preserving maps with a given covariant

Let Jac(u, v) be the Jacobian of a birational transformation in CP_2 : $K = C \cdot J$ where C is a collineation represented by a 3×3 matrix having determinant denoted by det(C).

B.1. Measure-preserving maps with covariant $\rho(u, v) = (u - 1)(v - 1)(u - v)$

The collineations which are such that the relation

$$\operatorname{Jac}(u, v) = \det(C) \frac{\rho(u', v')}{\rho(u, v)}$$

is satisfied with the covariant $\rho(u, v) = (u - 1)(v - 1)(u - v)$ can be grouped in three different classes, according to the eigenvalues of the associated matrices (see (B.2) and (B.4)). First, the collineation associated with matrix (7); second, those associated with the following three 3 × 3 matrices C_A , C_B , and C_C :

$$\begin{bmatrix} a_{31} - 1 & a_{32} & a_{33} + 1 \\ a_{31} - 1 & 1 + a_{32} & a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \begin{bmatrix} a_{31} & 1 + a_{32} & a_{33} - 2 \\ 1 + a_{31} & a_{32} & a_{33} - 2 \\ a_{31} & a_{32} & a_{33} - 1 \end{bmatrix}, \begin{bmatrix} 1 + a_{31} & -1 + a_{32} & a_{33} - 1 \\ a_{31} & -1 + a_{32} & a_{33} \\ a_{31} & a_{32} & a_{33} - 1 \end{bmatrix}$$
(B.1)

and third, those associated with the two 3×3 matrices C_D , and C_E :

$1 + a_{31}$	$-1 + a_{32}$	$a_{33} - 1$		a_{31}	$1 + a_{32}$	$a_{33} - 2$	
$1 + a_{31}$	a_{32}	$a_{33} - 2$,	$a_{31} - 1$	$1 + a_{32}$	$a_{33} - 1$	
a_{31}	a_{32}	$a_{33} - 1$		a_{31}	a_{32}	$a_{33} - 1$	

One easily verifies that all these matrices are stochastic-like matrices: the vector (1, 1, 1) is an eigenvector with a "stochastic" eigenvalue that we will denote as λ_{stoch} .

Let us first consider matrices C_A , C_B , and C_C . Their characteristic polynomials read:

$$P_A(t) = (t+1)(t-1)(t-a_{31}-a_{32}-a_{33}),$$

$$P_B(t) = P_C(t) = (t+1)(t-1)(t-a_{31}-a_{32}-a_{33}+1).$$
(B.2)

The "stochastic" eigenvalues λ_{stoch} of these three matrices (namely $a_{31} + a_{32} + a_{33}$ and $a_{31} + a_{32} + a_{33} - 1$) are the negative of the determinant of the corresponding matrices.

Actually, one can see that the three matrices C_A , C_B and C_C are simply related by row permutations combined with transformations on the a_{ij} 's parameters preserving their characteristic polynomials. Let us introduce the permutation matrix:

$$\mathcal{P} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let us also introduce the two matrices $\hat{C}_A(a_{31}, a_{32}, a_{33}) = C_A(1 + a_{31}, a_{32}, -2 + a_{33})$ and $\hat{C}_C(a_{31}, a_{32}, a_{33}) = C_C(a_{31}, 1 + a_{32}, -1 + a_{33})$, which have the same characteristic polynomial as C_B . One easily finds that

$$\mathcal{P}C_B = \hat{C}_A, \qquad \mathcal{P}^2 C_B = \hat{C}_C.$$

The three matrices C_A , C_B , and C_C are of the form C = H + P, where matrix H reads:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
(B.3)

for C_A , C_B , and C_C , respectively, and where matrix P becomes a projector when $\lambda_{\text{stoch}} = \pm 1$: $P^2 = 0$ if $\lambda_{\text{stoch}} = -\det(C) = +1$ and $P^2 + 2P = 0$ if $\lambda_{\text{stoch}} = -\det(C) = -1$.

When $\lambda_{\text{stoch}} = -\det(C) = \pm 1$, these matrices are *not* of finite order, as one could imagine from their characteristic polynomials in this $\lambda_{\text{stoch}} = -\det(C) = \pm 1$ limit.

Let us now consider matrices C_D and C_E . One sees that they are also equivalent up to a relabeling of the rows and columns: as far as the mappings are concerned, this corresponds to equivalence of the mappings up to transformations such as $(u, v) \rightarrow (v/u, 1/u)$.

The characteristic polynomials of C_D , C_E and (7) read, respectively, as

$$P_D(t) = P_E(t) = (t^2 - t + 1)(t + 1 - a_{31} - a_{32} - a_{33}), \qquad P_{(7)} = (t + 1)^2(t + 1 - a_{31} - a_{32} - a_{33}).$$
(B.4)

The "stochastic" eigenvalue λ_{stoch} of these three matrices (namely $a_{31}+a_{32}+a_{33}-1$) is also equal to the determinant of these matrices.

Let us consider $C = C_D$ or $C = C_E$. If one imposes $\det(C) = +\lambda_{\text{stoch}} = \pm 1$, then C is a matrix of order 6: $C_D^6 = C_E^6 = I_d$, where I_d denotes the 3×3 identity matrix. In the case $\det(C) = -1$ one even has $C_D^3 = C_E^3 = -I_d$.

Let us consider C to be (7). If one imposes det(C) = $+\lambda_{\text{stoch}} = 1$, then C is an involutive matrix: $C^2 = I_d$. If one imposes det(C) = $+\lambda_{\text{stoch}} = -1$, C is not a finite-order matrix, as one could imagine from its characteristic polynomial $(t + 1)^3$, it reads $C = -I_d + P$, where P is a projector: $P^2 = 0$. With this condition of equality of their determinant to +1, one easily finds that matrix (7) is an involution.

Note that these families of matrices are not families of commuting matrices, except if one restricts oneself to families depending on one parameter. Matrix (7) can be diagonalizes as $C = P\Delta P^{-1}$, where

$$P = \begin{bmatrix} 1 & -\frac{b}{a} & \frac{c}{a} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \Delta = \begin{bmatrix} a+b+c-1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

When one restricts oneself to one-parameter families and requires b/a and c/a to be constant (for instance, b = 5a, c = 7a), the associated matrices commute. Note, however, that, even in this case, the corresponding birational mappings do not commute.

B.2. Structure of mappings preserving the same covariant

Of course, if two birational mappings K_1 and K_2 share the same covariant $\rho(u, v)$, any product of these two mappings have the same covariant. Furthermore, if K_1 and K_2 both satisfy the measure-preserving necessary condition (11), any product of these two mappings also satisfy the measure-preserving necessary condition (11).

Let us consider the collineations of two variables u and v. The collineations that commute with the Hadamard inverse $(u, v) \rightarrow (1/u, 1/v)$ form a set of $24 = 4 \times 6$ transformations which are build from products of the four change-of-sign transformations $(u, v) \rightarrow (\pm u, \pm v)$, and the six permutations of the three rows and columns of the 3×3 matrix associated with the collineation, these six transformations being represented in u and v as (see also (10) in Section 2.1)

$$(u, v) \rightarrow \left(\frac{1}{u}, \frac{v}{u}\right), \left(\frac{1}{u}, \frac{u}{v}\right), (v, u), \left(\frac{u}{v}, \frac{1}{v}\right), (u, v), \left(\frac{v}{u}, \frac{1}{u}\right).$$

Let us denote by C_J one of these 24 collineations commuting with Hadamard inverse J. Considering the iteration of $K = C \cdot J$ it is straightforward to see that

$$K^N \cdot C_J = C_J \cdot \tilde{K}^N$$
, where $\tilde{K} = C_J^{-1} \cdot C \cdot C_J$. (B.5)

In other words, K and \tilde{K} have the same properties.

Suppose that C_J commutes with C, then it commutes with $K = C \cdot J$ and $K' = C \cdot C_J \cdot J = C_J \cdot K$ commute with K.

More generally, one can consider the set of two collineations C_1 and C_2 such that the "transmutation property" $C_1 \cdot J = J \cdot C_2$ holds. This problem is closely related to the commutation of the two transformations $K = C \cdot J$ and $K' = C' \cdot J$, which amounts to writing

$$C^{-1}C' \cdot J = J \cdot C'C^{-1}.$$
(B.6)

Appendix C. Recursion in one variable

For the CP₂ birational mapping (9), one can perform the elimination of v yielding a recursion on the successive u's (we denote u_n, u_{n+1} , and u_{n+2}):

$$K: (u_n, v_n) \to (u_{n+1}, v_{n+1}) = \left(\frac{au_n v_n + (b-1)v_n + cu_n}{(a-1)u_n v_n + bv_n + cu_n}, \frac{au_n v_n + bv_n + (c-1)u_n}{(a-1)u_n v_n + bv_n + cu_n}\right).$$
(C.1)

The elimination of v_n permits v_{n+1} to be expressed as a function of u_n and u_{n+1} as follows:

$$v_{n+1} = \frac{(b-1)u_n u_{n+1} - bu_{n+1} + au_n + 1 - a}{(a+b-2)(u_n-1)}.$$
(C.2)

Relation (C.1) is also valid shifting *n* by 1 and, thus, changing u_n into u_{n+1} , u_{n+1} into u_{n+2} , and v_n into v_{n+1} . Using (C.2) to eliminate v_{n+1} , one finds a recursion on the u_n 's

$$u_{n+2} = \frac{F_1(u_{n+1})u_n - F_2(u_{n+1})}{F_3(u_{n+1})u_n - F_4(u_{n+1})},$$
(C.3)

where the F_i 's are quadratic polynomials of u_{n+1} :

$$F_1(u_{n+1}) = a(b-1)u_{n+1}^2 - (2ab - 4a - 2b + 3)u_{n+1} + a(b-1),$$

$$F_2(u_{n+1}) = abu_{n+1}^2 - (2ab - 3a - 3b + 4)u_{n+1} + (a-1)(b-1),$$

$$F_3(u_{n+1}) = (a-1)(b-1)u_{n+1}^2 - (2ab - 3a - 3b + 4)u_{n+1} + ab$$

$$F_4(u_{n+1}) = (a-1)bu_{n+1}^2 - (2ab - 2a - 4b + 3)u_{n+1} + (a-1)b_{n+1}^2 - (a-1)b_$$

~

This recursion is reminiscent of the family of mappings of [71]. It has the same form as that of [71] where the F_i 's are linear.

Note that this elimination can still be performed for higher-dimensional birational mappings (see Section 4) but does *not* yield recursions such as (C.3) but, instead, algebraic relations between $u_n, u_{n+1}, u_{n+2}, \ldots, u_{n+p}$: $P(u_n, u_{n+1}, u_{n+2}, \ldots, u_{n+p}) = 0.$

Appendix D. Complexity analysis in CP₂ for mapping (9)

D.1. Growth-complexity from recursion (C.3)

The degree growth-complexity can be calculated from either the mapping (9) or from the recursion (C.3). The same singularity in the complexity generating functions appear in both cases. In this section, the iteration is described by the recursion (C.3) and the degrees of the numerators of the successive (bi)rational expressions are deduced accordingly.

• Complexity of (C.3) for a, b and c = 2 - a - b generic. Let us assume that neither a, b, nor c = 2 - a - b are of the form (N - 1)/N where N is a positive integer. The degree generating function deduced from recursion (C.3) reads:

$$G_{abc}(x) = \frac{x}{1 - 2x}.$$

• a = (N-1)/N, b and c = 2 - a - b generic. Let us consider a = (N-1)/N and b in (C.3), parameter b being arbitrary. The degree generating function reads:

$$G_{Nbc}(x) = -x^{N+1} + \frac{x(1-x)}{1-2x+x^{N+1}} = -x^{N+1} + \frac{x}{1-x-x^2-x^3-\dots-x^N}$$

• a = (N-1)/N, b = (M-1)/M, c = 2 - a - b generic. Let us consider a = (N-1)/N and b = (M-1)/M in (C.3), where N > M.

Let us first consider M = 3, i.e., b = 2/3. The degree generating function reads:

$$G_{N3c}(x) = -x^{N+1} + \frac{x(1-x)}{1-2x+x^4+x^{N+1}-x^{N+3}} = -x^{N+1} + \frac{x}{1-x-x^2-x^3+x^{N+1}(1+x)}$$

for generic N except for N = 2, 3 and 6, for which the birational mapping becomes integrable.

Actually, for these values, the associated value of c = 2 - a - b becomes of the singled-out form (P - 1)/P, where *P* is an integer and the degree generating function reads, for instance, for a = 5/6 and b = 2/3

$$G_{63c}(x) = -x^7 + \frac{x(1+x^2)(1+x^4)}{(1+x+x^2)(1+x+x^2+x^3+x^4)(1-x)^3}$$

with the denominator having zeros only at Nth roots of unity (polynomial growth).

Let us now consider M = 4, i.e., b = 3/4. The degree generating function reads:

$$G_{N4c}(x) = -x^{N+1} + \frac{x(1-x)}{1-2x+x^5+x^{N+1}-x^{N+4}} = -x^{N+1} + \frac{x}{1-x-x^2-x^3-x^4+x^{N+1}(1+x+x^2)}$$

for generic N except for N = 2 and 4, for which the mapping becomes integrable.

For instance, for N = 4, i.e., a = b = 3/4, one has

$$G_{44c}(x) = \frac{x(x^3 - x + 1)}{(1 + x^4)(1 + x)(1 - x)^3}$$

For arbitrary (N, M) values, with N > M, one conjectures the following generating function:

$$G_{NMc}(x) = -x^{N+1} + \frac{x(1-x)}{1-2x+x^{M+1}+x^{N+1}-x^{N+M}}$$

= $-x^{N+1} + \frac{x}{1-x-x^2-x^3-\dots-x^M+x^{N+1}(1+x+\dots+x^{M-2})}.$

D.2. Dynamical zeta function for (9) for M even

For a = 1/2 (namely M = 2), b and c = 2 - a - b generic, the expansion of the dynamical zeta function, obtained up to order 13, is compatible with a $1 - x - x^2$ singularity for the dynamical zeta function $\zeta_{2,b,c}(x)$

$$\rho_{2,b,c}(x) = (1 - x - x^2)\zeta_{2,b,c}(x)$$

= $(1 - x^3)(1 - x^5)^2(1 - x^7)^4(1 - x^8)(1 - x^9)^6(1 - x^{10})(1 - x^{11})^{12}(1 - x^{12})^3(1 - x^{13})^{20}\cdots$

Actually, the expansion of $\rho_{2,b,c}(x)$ corresponds to coefficients that do *not* grow exponentially. For instance,

$$\rho_{2,b,c}(x) = 1 - x^3 - 2x^5 - 4x^7 + x^8 - 6x^9 + 4x^{10} - 11x^{11} + 11x^{12} - 18x^{13} + \cdots$$

Unfortunately, we do not have a large enough expansion to see if $\rho_{2,b,c}(x)$, is a actually a *rational* expression with a denominator with zeros only at *N*th root of unity.

Similarly, for a = 3/4 (namely M = 4), b and c = 2 - a - b generic, the expansion of the dynamical zeta function, obtained up to order 11, is compatible with a $1 - x - x^2 - x^3 - x^4$ singularity for the corresponding dynamical zeta function $\zeta_{4,b,c}(x)$:

$$\rho_{4,b,c}(x) = (1 - x - x^2 - x^3 - x^4)\zeta_{4,b,c}(x) = (1 - x^5)(1 - x^7)^2(1 - x^9)^4(1 - x^{11})^8 \cdots$$

For a = 5/6 (namely M = 6), b and c = 2 - a - b generic, the expansion, up to order 9, is compatible with a $1 - x - x^2 - x^3 - x^4 - x^5 - x^6$ singularity

$$\zeta_{6,b,c}(x) = (1 - x - x^2 - x^3 - x^4 - x^5 - x^6)\zeta_{6,b,c}(x) = (1 - x^7)(1 - x^9)^2 \cdots$$

Similarly, the expansions of these ρ 's correspond to coefficients that do not seem to grow exponentially.

Appendix E. Growth-complexity for CP₃

In this appendix, we consider semi-numerical iterations of mapping (34) for various values of the parameters. This will lead us to evaluate the growth-complexity λ in each case. These calculations can be done either numerically, i.e., by taking a numerical *rational* initial point, or semi-numerically, i.e., by taking an initial point of the form $(u(t), v(t), w(t)) = (\alpha_u t + \beta_u, \alpha_v t + \beta_v, \alpha_w t + \beta_w)$, where t is a parameter and the α 's and β 's are integers (this is equivalent to iterating a parameterized line) and considering generating functions of the degree in t of the numerators of successive u(t)'s (resp. v(t)'s or w(t)'s). Because the growth-complexity λ is a topological invariant [27,28,30,31], the denominators of these (rational) degree generating functions are independent of the choice of

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the line $(\alpha_u t + \beta_u, \alpha_v t + \beta_v, \alpha_w t + \beta_w)$ that one iterates, the numerator being a slightly less universal quantity. To obtain more universal degree generating functions, the calculation must be performed on the mapping written homogeneously in homogeneous variables [28]; however, this yields much larger formal calculations.

Numerically, one can have, rapidly, a very good approximation of the growth-complexity λ by iterating a rational number, considering u_i after simplifications, namely N_i/D_i (N_i and D_i are integers). The ratio of the number of the digits of two successive numerators N_{i+1} and N_i (resp. two successive denominators D_{i+1} and D_i) is a good approximation [27] of complexity λ .

Let us show, more explicitly, how the method works in the case of our mapping (34) for various subcases of the parameters of the mapping.

E.1. Complexity of (34) for a, b, c and d = 2 - a - b - c generic

Let us assume that neither a, b, c, nor d = 2 - a - b - c are of the form (N - 1)/N, where N is a positive integer. (We will call such a situation "generic".) In this case, the degree generating function is always compatible, up to order 7, with

$$G_{abc}(x) = \frac{x}{1 - 3x}.$$

E.2. Complexity of (34) for a = (N-1)/N and b, c, d = 2 - a - b - c, generic

Let us consider a = (N-1)/N in (34) and parameters b, c, d = 2 - a - b - c being generic ($\neq (N-1)/N$).

• The calculations corresponding to the iteration of the line (u(t), v(t), w(t)) = (3t+2, 6t+5, -t+7), for instance, have been performed up to order 7 for N = 2, i.e., a = 1/2. The generating function of the degree of t in the numerators of u(t) has the expansion:

$$G_{2bc}^{(1)}(x) = 3x + 8x^2 + 25x^3 + 69x^4 + 189x^5 + 517x^6 + 1413x^7 + 3861x^8 + \cdots,$$

which is compatible with

$$G_{2bc}^{(1)}(x) = \frac{x(3-x+x^2-2x^4)}{1-3x+2x^3} = -1 - x^2 + \frac{1}{1-3x+2x^3}.$$

Similarly, for the same values of *a*, *b*, *c*, and *d*, the calculations corresponding to the iteration of another line, for instance, the line (u(t), v(t), w(t)) = (t, 11, 13), have been performed up to order 6 for N = 2, i.e., a = 1/2. The generating function of the degree of *t* in the numerators of u(t) has the expansion:

$$G_{2bc}^{(2)}(x) = x + 3x^2 + 9x^3 + 25x^4 + 69x^5 + 189x^6 + 517x^7 + 1413x^8 + \cdots,$$

which is compatible with

$$G_{2bc}^{(2)}(x) = \frac{x}{1 - 3x + 2x^3}$$

The two methods give the same singularity $(1 - 3x + 2x^3)$, as is expected, corresponding to the complexity $\lambda \simeq 2.73205$.

• For a = 2/3, one obtains similarly, iterating, for instance, line (u(t), v(t), w(t)) = (t, 11, 13):

$$G_{3bc}(x) = x + 3x^2 + 9x^3 + 27x^4 + 79x^5 + 231x^6 + 675x^7 + 1971x^8 + \cdots$$

corresponding to the following degree generating function:

$$G_{3bc}(x) = \frac{x}{1 - 3x + 2x^4}$$

The singularity obtained corresponds to the complexity $\lambda\simeq 2.91964.$

• For a = 3/4, one obtains:

$$G_{4bc}(x) = x + 3x^2 + 9x^3 + 27x^4 + 81x^5 + 241x^6 + 717x^7 + 2133x^8 + \cdots$$

corresponding to the following degree generating function:

$$G_{4bc}(x) = \frac{x}{1 - 3x + 2x^5}.$$

The singularity obtained corresponds to the complexity $\lambda \simeq 2.97445$.

We conjecture that the denominators of the generating functions should be for arbitrary N: $1 - 3x + 2x^{N+1}$.

E.3. Complexity of (34) for a = b = (N - 1)/N and c, d = 2 - a - b - c generic

For a = b = 1/2, one obtains similarly, iterating, for instance, the line (u(t), v(t), w(t)) = (t, 11, 13):

$$G_{22c}(x) = x + 3x^2 + 7x^3 + 17x^4 + 41x^5 + 99x^6 + 239x^7 + 577x^8 + 1393x^9 + \cdots$$

corresponding to the following degree generating function:

$$G_{22c}(x) = \frac{x(1-2x^2+x^4)}{1-3x-x^4-x^5+4x^3} = -\frac{x}{1+x} + \frac{2x}{(1+x)(1-2x-x^2)}.$$

The singularity obtained corresponds to the complexity $\lambda \simeq 2.41421$.

For a = b = 2/3, one has the following expansion for the degree generating function:

$$G_{33c}(x) = x + 3x^2 + 9x^3 + 25x^4 + 71x^5 + 201x^6 + 569x^7 + 1611x^8 + \cdots$$

corresponding to

$$G_{33c}(x) = \frac{x(1-2x^3+x^6)}{1-3x-x^6-x^7+4x^4} = -\frac{x}{1+x} + \frac{2x}{(1+x)(1-2x-2x^2-x^3)}.$$

The singularity obtained corresponds to the complexity $\lambda \simeq 2.83118$.

For a = b = 3/4, one deduces the following expansion for the degree generating function:

$$G_{44c}(x) = x + 3x^2 + 9x^3 + 27x^4 + 79x^5 + 233x^6 + 687x^7 + 2025x^8 + \cdots$$

corresponding to

$$G_{44c}(x) = \frac{x(1-2x^4+x^8)}{1-3x+4x^5-x^8-x^9} = -\frac{x}{1+x} + \frac{2x}{(1+x)(1-2x-2x^2-2x^3-x^4)}.$$

The singularity obtained corresponds to the complexity $\lambda \simeq 2.94771$.

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E.4. Complexity of (34) for a = (N - 1)/N, b = (M - 1)/M and c, d = 2 - a - b - c generic

Let us consider a = (N - 1)/N, b = (M - 1)/M with the simplest case N = 2 and M = 3. The degree generating function reads, when iterating, for instance, line (u(t), v(t), w(t)) = (t, 11, 13):

$$G_{23c}(x) = x + 3x^2 + 9x^3 + 23x^4 + 61x^5 + 161x^6 + 423x^7 + 1113x^8 + \cdots,$$

which is compatible with

$$G_{23c}(x) = \frac{x(1-2x^3+x^5)}{1-3x+2x^3+2x^4-x^5-x^6} = -\frac{x}{1+x} + \frac{2x}{(1+x)(1-x)(1-x-3x^2-3x^3-x^4)}$$

The singularity obtained corresponds to the complexity $\lambda \simeq 2.62966$.

For N = 3 and M = 4, the degree generating function reads:

$$G_{34c}(x) = x + 3x^2 + 9x^3 + 27x^4 + 77x^5 + 223x^6 + 645x^7 + 1865x^8 + \cdots,$$

which is compatible with

$$G_{34c}(x) = \frac{x(1-2x^4+x^7)}{1-3x+2x^4+2x^5-x^7-x^8}$$

= $-\frac{x}{1+x} + \frac{2x}{(1+x)(1-x)(1-x-3x^2-5x^3-5x^4-3x^5-x^6)}$

The singularity obtained corresponds to the complexity $\lambda \simeq 2.89089$.

For N = 2 and M = 4, the degree generating function reads:

$$G_{24c}(x) = x + 3x^2 + 9x^3 + 25x^4 + 67x^5 + 181x^6 + 489x^7 + 1319x^8 + \cdots,$$

which is compatible with

$$G_{24c}(x) = \frac{x(1-2x^4+x^6)}{1-3x+2x^3+2x^5-x^6-x^7} = -\frac{x}{1+x} + \frac{2x(1-x+x^2)}{(1+x)(1-x)(1-2x-x^2-2x^3-x^4)}$$

The singularity obtained corresponds to the complexity $\lambda \simeq 2.69679$.

For N = 2 and M = 5, the degree generating function reads:

$$G_{25c}(x) = x + 3x^2 + 9x^3 + 25x^4 + 69x^5 + 187x^6 + 509x^7 + 1385x^8 + \cdots,$$

which is compatible with

$$G_{25c}(x) = \frac{x(1-2x^5+x^7)}{1-3x+2x^3+2x^6-x^7-x^8}$$

= $-\frac{x}{1+x} + 2\frac{x(x^4+x^3+1)}{(1+x)(1-x)(1-x-3x^2-3x^3-3x^4-3x^5-x^6)}.$

The singularity obtained corresponds to the growth-complexity $\lambda \simeq 2.69679$.

We conjecture that, when *a* and *b* are of the form a = (N-1)/N and b = (M-1)/M, the denominators of the generating functions should be for arbitrary *N* and *M*: $1 - 3x + 2x^{M+1} + 2x^{N+1} - x^{M+N} - x^{M+N+1}$.

E.4.1. Complexity of (34) for a = b = (N - 1)/N, c = (P - 1)/P and d = 2 - a - b - c generic

Let us consider a = b = (N - 1)/N, c = (P - 1)/P. In the simplest case, N = M = 2 and P = 3, for an iteration of, for instance, line (u(t), v(t), w(t)) = (3t + 2, 6t + 5, -t + 7), the generating function of the degree of the numerators of the u(t)'s yields, up to K^9 , the expansion

 $G_{223}(x) = 3x + 8x^2 + 22x^3 + 52x^4 + 120x^5 + 274x^6 + 624x^7 + 1418x^8 + 3220x^9 + \cdots$

compatible with the rational expression

$$G_{223}(x) = -1 - x^2 + \frac{(1+x^2)(1+x+x^2)}{(1-x)(1-x-2x^2-2x^3)} = \frac{x(1+x^2)(3+2x-2x^3)}{(1-x)(1-x-2x^2-2x^3)}.$$

The singularity obtained corresponds to the complexity $\lambda \simeq 2.269531$.

Let us also remark that numerical calculations with rational numbers, as explained before, have been performed, up to K^{15} , yielding a growth $\lambda \simeq 2.269518$ compatible with the singularity $1 - 2x - x^2 + 2x^4$.

E.5. Complexity of (34) for a, b, c, d = 2 - a - b - c of the form (N - 1)/N

All of these are integrable cases. One obtains the solutions already obtained for CP_2 . There is however one new solution (2, 2, 2, 2) (see (C.3)). This is completely in agreement with the Diller–Favre conditions [34] which has been proved for CP_2 and seem, also, to apply for our particular mappings of CP_n .

References

- [1] E. Bedford, On the interplay between real and complex dynamical systems (dedicated to the memory of K. Oka).
- [2] M.P. Bellon, J.-M. Maillard, C.-M. Viallet, Integrable coxeter groups, Phys. Lett. A 159 (1991) 221–232.
- [3] M.P. Bellon, J.-M. Maillard, C.-M. Viallet, Higher dimensional mappings, Phys. Lett. A 159 (1991) 233-244.
- [4] M.P. Bellon, J.-M. Maillard, C.-M. Viallet, Infinite discrete symmetry group for the Yang–Baxter equations: spin models, Phys. Lett. A 157 (1991) 343–353.
- [5] M.P. Bellon, J.-M. Maillard, C.-M. Viallet, Infinite discrete symmetry group for the Yang–Baxter equations: vertex models, Phys. Lett. B 260 (1991) 87–100.
- [6] M.P. Bellon, J.-M. Maillard, C.-M. Viallet, Rational mappings, arborescent iterations, and the symmetries of integrability, Phys. Rev. Lett. 67 (1991) 1373–1376.
- [7] M.P. Bellon, J.-M. Maillard, C.-M. Viallet, Quasi-integrability of the sixteen-vertex model, Phys. Lett. B 281 (1992) 315–319.
- [8] M.P. Bellon, J.-M. Maillard, C.-M. Viallet, Dynamical systems from quantum integrability, in: J.-M. Maillard (Ed.), Proceedings of the Conference Yang–Baxter Equations in Paris, World Scientific, Singapore, 1993, pp. 95–124, Int. J. Mod. Phys.
- [9] C.-M. Viallet, On some rational Coxeter groups, CRM Proceedings and Lecture Notes, vol. 9, 1996, pp. 377–388.
- [10] S. Boukraa, J.-M. Maillard, Let's Baxterise, J. Stat. Phys. 102 (2001) 641–700. hep-th/0003212.
- [11] J.-M. Maillard, Automorphisms of algebraic varieties and Yang-Baxter equations, J. Math. Phys. 27 (1986) 2776.
- [12] S. Boukraa, J.-M. Maillard, G. Rollet, Discrete symmetry groups of vertex models in statistical mechanics, J. Stat. Phys. 78 (1995) 1195–1251.
- [13] S. Boukraa, J.-M. Maillard, G. Rollet, Almost integrable mappings, Int. J. Mod. Phys. B 8 (1994) 137–174.
- [14] S. Boukraa, J.-M. Maillard, G. Rollet, Integrable mappings and polynomial growth, Physica A 209 (1994) 162–222.
- [15] S. Boukraa, J.-M. Maillard, G. Rollet, Determinantal identities on integrable mappings, Int. J. Mod. Phys. B 8 (1994) 2157–2201.
- [16] S. Boukraa, S. Hassani, J.-M. Maillard, New integrable cases of a Cremona transformation: a finite-order orbits analysis, Physica A 240 (1997) 586–621.
- [17] H. Meyer, J.-C. Anglès d'Auriac, J.-M. Maillard, G. Rollet, Phase diagram of a six-state chiral Potts model, Physica A 208 (1994) 223-236.
- [18] M.P. Bellon, J.-M. Maillard, G. Rollet, C.-M. Viallet, Deformations of dynamics associated to the chiral Potts model, Int. J. Mod. Phys. B 6 (1992) 3575–3584.
- [19] M. Noether, Über Flächen welche Schaaren rationaler Curven besitzen, Math. Ann. 3 (1871) 161–227.
- [20] M. Noether, Luigi Cremona, Mathematische Annalen 59 (1904) 1-19.
- [21] Alexander, On the factorization of Cremona plane transformations, Trans. Am. Math. Soc. 17 (1916) 295–300.

- [22] Algebraic surfaces, K. Kodaira, D.C. Spencer (Eds.), Proceedings of the Seminar of I.R. Šafarevič, No. 75, Steklov Institute of Mathematics, American Mathematical Society, Providence, 1965, pp. 14–20.
- [23] N.L. Biggs, Math. Proc. Cambridge, Philos. Soc. 80, 429.
- [24] N.L. Biggs, Interaction Models, Cambridge University Press, Cambridge, 1977.
- [25] D. Hansel, J.-M. Maillard, Symmetries of models with genus >1, Phys. Lett. A 133 (1988) 11-15.
- [26] N. Abarenkova, J.-C. Anglès d'Auriac, S. Boukraa, S. Hassani, J.-M. Maillard, Rational dynamical zeta functions for birational transformations, Physica A 264 (1999) 264–293.
- [27] N. Abarenkova, J.-C. Anglès d'Auriac, S. Boukraa, S. Hassani, J.-M. Maillard, Topological entropy and complexity for discrete dynamical systems, Phys. Lett. A 262 (1999) 44–49. chao-dyn/9806026.
- [28] N. Abarenkova, J.-C. Anglès d'Auriac, S. Boukraa, S. Hassani, J.-M. Maillard, Real Arnold complexity versus real topological entropy for birational transformations, J. Phys. A 33 (2000) 1465–1501. chao-dyn/9906010.
- [29] N. Abarenkova, J.-C. Anglès d'Auriac, S. Boukraa, J.-M. Maillard, Growth-complexity spectrum of some discrete dynamical systems, Physica D 130 (1999) 27–42. chao-dyn/9807031.
- [30] V. Arnold, in: V. Arnold, M. Monarstyrsky (Eds.), Problems on Singularities and Dynamical Systems, Developments in Mathematics, The Moscow School, Chapman & Hall, London, 1993.
- [31] V.I. Arnold, Dynamics of complexity of intersections, Bol. Soc. Bras. Mat. 21 (1990) 1-10.
- [32] N. Abarenkova, J.-C. Anglès d'Auriac, S. Boukraa, J.-M. Maillard, Real topological entropy versus metric entropy for birational measure-preserving transformations, Physica D 144 (2000) 387–433.
- [33] C. Favre, Points périodiques d'applications birationnelles de P², Ann. Inst. Fourier (Grenoble) 48 (4) (1998) 999–1023.
- [34] J. Diller, C. Favre, Dynamics of bimeromorphic maps of surfaces, Am. J. Math. 123 (6) (2001) 1135–1169.
- [35] H. Sakai, Rational surfaces associated with affine root systems and geometry of the Painlevé equations, 1999, Preprint. nlin.SI/0103016.
- [36] T. Takenawa, Discrete dynamical systems associated with root systems of indefinite type, 2001, Preprint. nlin.SI/0103016.
- [37] T. Takenawa, A geometric approach to singularity confinement and algebraic entropy. J. Phys. A 3 (2001) L95, 2000, Preprint. nlin.SI/0011037.
- [38] J. Hietarinta, C.-M. Viallet, Singularity confinement and chaos in discrete systems, Phys. Rev. Lett. 81 (1998) 325.
- [39] M. Menghini, Notes on the correspondence between Luigi Cremona and Max Noether, Historia Mathematica 13 (4) (1986) 341-351.
- [40] J.L.A. Coolidge, Treatise on Algebraic Plane Curves, Dover, New York, 1959, p. 207.
- [41] J.L.A. Coolidge, A History of Geometrical Methods, Dover, New York, 1963 (reprint).
- [42] P. Griffiths, J. Harris, Principles of Algebraic Geometry, Wiley, New York, 1978, pp. 497–498.
- [43] A. Corti, Factoring birational maps of threefolds after Sarkisov, J. Alg. Geom. 4 (1995) 223-254.
- [44] V.A. Iskovskikh, On the rationality problem for three-dimensional algebraic varieties, Proc. Steklov Inst. Math. 218 (1997) 185-227.
- [45] A.V. Pukhlikov, Birational automorphisms of four-dimensional quintics, Inv. Math. 87 (1987) 303–329.
- [46] A.V. Pukhlikov, Birational automorphisms of higher-dimensional varieties, Documenta Mathematica Extra Volume ICM, 1998, II-97-107.
- [47] A.V. Pukhlikov, Birational automorphisms of three-dimensional quartic with a quadratic singularity, Math. USSR Sbornik 66 (1990) 265–284.
- [48] A. Corti, Recent results in higher-dimensional birational geometry, Complex Algebraic Geometry, MSRI Publications, vol. 28, 1995, pp. 35–56.
- [49] V.G. Sarkisov, Birational maps of standard Q-Fano fiberings, I.V. Kurchatov Institute Atomic Energy, 1989, Preprint.
- [50] V.G. Sarkisov, Birational automorphisms of conic bundles, Math. USSR Izv. 17 (1981) 355–390.
- [51] S. Boukraa, J.-M. Maillard, Factorization properties of birational mappings, Physica A 220 (1995) 403-470.
- [52] G.R.W. Quispel, J.A.G. Roberts, Reversible mappings of the plane, Phys. Lett. A 132 (1988) 161–163.
- [53] N. Abarenkova, J.-C. Anglès d'Auriac, S. Boukraa, S. Hassani, J.-M. Maillard, From Yang–Baxter equations to dynamical zeta functions for birational transformations, in: L.T. Wille, M. Batchelor (Eds.), Statistical Mechanics on the Eve of the 21st Century, World Scientific, Singapore, 1999.
- [54] M. Artin, B. Mazur, On periodic points, Ann. Math. 81 (1965) 82.
- [55] J.-C. Anglès d'Auriac, J.-M. Maillard, C.-M. Viallet, A classification of four-state spin edge models, J. Phys. A 35 (2002) 9251–9272. cond-mat/0209557.
- [56] R. Hartshorne, Algebraic geometry, Graduate Texts in Mathematics, Springer, Berlin, 1977, Chapter C.
- [57] H. Hironaka, Resolution of singularities of an algebraic variety over field of characteristic zero, Ann. Math. 79 (2) (1964) 109–203;
 H. Hironaka, Resolution of singularities of an algebraic variety over field of characteristic zero, Ann. Math. 79 (2) (1964) 205–326.
- [58] O. Zariski, Collected Papers, vol. I-IV, MIT Press, Cambridge, 1972-1979.
- [59] L. Bonavero, Factorisation faible des applications birationnelles, Séminaire Bourbaki, Numéro 880, 53-ième année (2000–2001).
- [60] J. Wlodarczyk, Birational cobordism and factorisation of birational maps, J. Alg. Geom. 9 (3) (2000) 425-449.
- [61] Yu.I. Manin, Rational surfaces over perfect fields, Inst. Hautes Etudes Sci. Publ. Math. 30 (1966) 56-97.
- [62] Yu.I. Manin, Rational surfaces over perfect fields. II, Mat. Sbornik 72 (1967) 161-192.
- [63] V.A. Iskovskikh, Yu.I. Manin, Three-dimensional quartics and counterexamples to the Lüroth problem, Math. USSR Sb. 15 (1) (1971) 141–166.
- [64] V.A. Iskovskikh, Birational automorphisms of three-dimensional algebraic varieties, J. Soviet Math. 13 (1980) 815-868.

- [65] V.A. Iskovskikh, Algebraic threefolds with a special regard to the problem of rationality, in: Proceedings of the International Congress on Mathematics, Warsawa, 1983, PWM, Warsawa, 1984, pp. 733–746.
- [66] J. Kollar, Nonrational covers of $CP_n \times CP_n$, Proc. Symp. Alg. Geom., in press. alg-geom/g606005.
- [67] J. Kollar, et al., Flips and abundance for algebraic threefolds, Seminar Proceedings, Utah, August 1991 (Astérisque 211).
- [68] A. Corti, Singularities of linear systems and 3-fold birational geometry, in: A. Corti, M. Reid (Eds.), Explicit Birational Geometry of 3-folds, LMS-LNS, vol. 281, 2000, pp. 259–312.
- [69] J. Kollar, S. Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998.
- [70] S. Mori, Classification of higher-dimensional varieties, Proc. Symp. Pure Math. 46 (1987) 269-331.
- [71] J.A.G. Roberts, G.R.W. Quispel, Chaos and time-reversal symmetry: order and chaos in reversible dynamical systems, Phys. Rep. 216 (1992) 63–177.
- [72] M. Gizatullin, On some tensor representations of the Cremona group of the projective plane, in: K. Hulek, M. Reid, C. Peters, F. Catanese (Eds.), Warwick Euroconference, July 1996, New Trends in Algebraic Geometry, Cambridge University Press. alg-geom/970600.
- [73] I. Pan, F. Ronga, T. Vust, Transformations birationnelles quadratiques de l'espace projectif complexe à trios dimensions, Ann. Inst. Fourier 51 (2001) 1153–1187.