Nonperturbative Functional Renormalization-Group Approach to the Sine-Gordon Model and the Lukyanov-Zamolodchikov Conjecture

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(Received 10 December 2018; published 19 April 2019)

We study the quantum sine-Gordon model within a nonperturbative functional renormalization-group approach (FRG). This approach is benchmarked by comparing our findings for the soliton and lightest breather (soliton-antisoliton bound state) masses to exact results. We then examine the validity of the Lukyanov-Zamolodchikov conjecture for the expectation value $\langle e^{(i/2)n\beta\varphi}\rangle$ of the exponential fields in the massive phase (n is integer and $2\pi/\beta$ denotes the periodicity of the potential in the sine-Gordon model). We find that the minimum of the relative and absolute disagreements between the FRG results and the conjecture is smaller than 0.01.

DOI: 10.1103/PhysRevLett.122.155301

Introduction.—The quantum sine-Gordon model [1] describes many physical systems. In condensed matter it is widely used to understand the phase diagram and the low-energy properties of one-dimensional quantum fluids [2–4] and has applications that range from strongly correlated electron systems to cold atoms. In high-energy physics, it is related to the massive Thirring model describing Dirac fermions with a self-interaction [5]. The sine-Gordon model can also be viewed as a two-dimensional model of classical statistical mechanics. In particular it describes the Berezinskii-Kosterlitz-Thouless (BKT) transition that occurs in the XY spin model and more generally in two-dimensional systems with a two-component order parameter with an O(2) symmetry [6–8].

The Hamiltonian of the quantum sine-Gordon model is defined by

$$\hat{H} = \int dx \left\{ \frac{1}{2}\hat{\Pi}^2 + \frac{1}{2}\left(\frac{\partial\hat{\varphi}}{\partial x}\right)^2 - u\cos(\beta\hat{\varphi}) \right\}, \qquad (1)$$

where $\hat{\Pi}$ and $\hat{\varphi}$ satisfy canonical commutation relations, $[\hat{\varphi}(x), \hat{\Pi}(x')] = i\delta(x-x')$. Regularization with a UV momentum cutoff Λ is implied and u/Λ^2, $\beta > 0$ are dimensionless parameters. The phase diagram consists of a gapless phase with massless (anti)soliton excitations for $\beta^2 \geq 8\pi$ (and $u \to 0$) and a gapped phase with massive (anti) soliton excitations for $\beta^2 \leq 8\pi$. The soliton and the anti-soliton carry the topological charge $Q = 1$ and -1, respectively [9]. They attract for $\beta^2 \leq 4\pi$ and can form bound states, called breathers, with topological charge $Q = 0$. The phase transition between the two phases is of BKT type [6–8].

The sine-Gordon model is one of the most studied integrable models; its spectrum, thermodynamics, and scattering properties are well understood [10–14]. However not everything is known and many quantities can be obtained only from nonexact (e.g., perturbative) methods [2–4]. In particular, in the massive phase, the amplitude of the fluctuations about the mean value $\langle\hat{\varphi}\rangle = 0$ is not known exactly. It has been conjectured by Lukyanov and Zamolodchikov that [15]

$$\langle e^{i\sqrt{8\pi}a\hat{\varphi}}\rangle = \left(\frac{\Gamma(1-K)}{\Gamma(K)}\frac{\pi u}{2(b\Lambda)^2}\right)^{a^2/(1-K)} \exp\left[\int_0^\infty \frac{dt}{t}\left(\frac{\sinh^2(2a\sqrt{K}t)}{2\sinh(Kt)\sinh(t)\cosh[(1-K)t]} - 2a^2 e^{-2t}\right)\right], \qquad (2)$$

where $|\Re(a)| < 1/2\sqrt{K}$ and $K = \beta^2/8\pi$ is the "Luttinger parameter" [4] (the massive phase corresponds to $K < 1$). Equation (2) is exact for $a = \sqrt{K}$, $K = 1/2$, and in the semiclassical limit $K \to 0$ [16]. Additional arguments supporting the conjecture were presented in Refs. [17,18]. From the equivalence between the sine-Gordon model and the massive Thirring model, Eq. (2) was shown to be correct to first order in u [19,20]. Further evidence of the correctness of Eq. (2), in particular for not too large values of a, was provided by a numerical study in a finite volume [21] and variational perturbation theory [22].

In this Letter, we examine the validity of the Lukyanov-Zamolodchikov conjecture using a nonperturbative functional renormalization-group (FRG) approach [23,24]. We go beyond previous FRG approaches [25–28] and, in order to benchmark our approach, first compute the mass $M_{\rm sol}$ of the (anti)soliton as well as that (M_1) of the lightest breather. We then turn to the computation of the expectation value $\langle e^{(i/2)n\beta\hat{\varphi}}\rangle = \langle e^{in\sqrt{2\pi K}\hat{\varphi}}\rangle$ (n integer) of the exponential fields. We confirm the Lukyanov-Zamolodchikov conjecture with an accuracy, defined as the minimum of the relative and absolute disagreements between the FRG results and the conjecture, of 0.01.

FRG approach.—From now on we adopt the point of view of classical statistical mechanics (or Euclidean field theory) where the sine-Gordon model is defined by the partition function

$$\mathcal{Z}[J] = \int \mathcal{D}[\varphi] e^{-\int d^2r\{(1/2)(\nabla\varphi)^2 - u\cos(\beta\varphi) - J\varphi\}}, \quad (3)$$

with $\varphi(\mathbf{r})$ being a classical field and $\mathbf{r}$ being a two-dimensional coordinate. J is an external source allowing us to obtain the expectation value $\phi(\mathbf{r}) = \langle\varphi(\mathbf{r})\rangle = \delta\ln\mathcal{Z}[J]/\delta J(\mathbf{r})$ by functional derivation. Most physical quantities can be obtained from the free energy $-\ln\mathcal{Z}[J]$ or, equivalently, the effective action (or Gibbs free energy)

$$\Gamma[\phi] = -\ln\mathcal{Z}[J] + \int d^2r J\phi, \quad (4)$$

defined as the Legendre transform of $\ln\mathcal{Z}[J]$.

We compute $\Gamma[\phi]$ using a Wilsonian nonperturbative FRG approach where fluctuation modes are progressively integrated out in the functional integral [Eq. (3)]. This defines a scale-dependent effective action $\Gamma_k[\phi]$, which incorporates fluctuations with momenta between a (running) momentum scale k and the UV scale $k_{\rm in} \gg \Lambda$. The latter condition implies that the initial value $\Gamma_{k_{\rm in}}[\phi] = S[\phi]$ coincides, as in mean-field theory, with the microscopic action defined by Eq. (3). The effective action of the sine-Gordon model, $\Gamma_{k=0}[\phi]$, is obtained when all fluctuations have been integrated out. The scale-dependent effective action satisfies an exact flow equation that cannot be solved exactly [29]. A common approximation scheme is the derivative expansion where

$$\Gamma_k[\phi] = \int d^2r\left\{\frac{1}{2}Z_k(\phi)(\nabla\phi)^2 + U_k(\phi)\right\} \quad (5)$$

is truncated to second order in derivatives. This leads to coupled flow equations for the functions $Z_k(\phi)$ and $U_k(\phi)$, with initial conditions $Z_{k_{\rm in}}(\phi) = 1$ and $U_{k_{\rm in}}(\phi) = -u\cos(\beta\phi)$, which can be solved numerically. We refer to the Supplemental Material for more detail about the implementation of the FRG approach [30].

It is convenient to consider the dimensionless functions

$$\tilde{Z}_k(\phi) = \frac{Z_k(\phi)}{Z_k}, \qquad \tilde{U}_k(\phi) = \frac{U_k(\phi)}{Z_k k^2}, \quad (6)$$

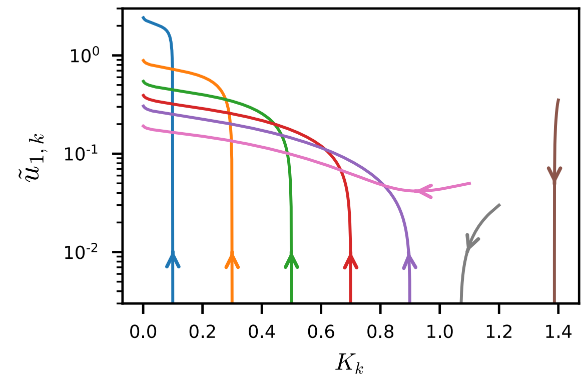

FIG. 1. Flow diagram of the sine-Gordon model projected onto the plane $(K_k, \tilde{u}_{1,k})$, where $\tilde{u}_{1,k}$ is the first harmonic of the potential $\tilde{U}_k(\phi)$ and K_k is the running Luttinger parameter. There is an attractive line of fixed points for $\tilde{u}_{1,k} = 0$ and $K_k > 1$ that terminates at the BKT point ($\tilde{u}_{1,k} = 0, K_k = 1$).

where $Z_k = \langle Z_k(\phi)\rangle_\phi$ is obtained by averaging $Z_k(\phi)$ on $[-\beta/\pi, \beta/\pi]$ [35]. The flow diagram, projected onto the plane $(K_k, \tilde{u}_{1,k})$, is shown in Fig. 1. Here $\tilde{u}_{1,k}$ is the first harmonic of the potential $\tilde{U}_k(\phi) = -\sum_{n=0}^{\infty}\tilde{u}_{n,k}\cos(n\beta\phi)$ and $K_k = K/Z_k$ can be interpreted as a "running" Luttinger parameter. In the massive phase, the flow runs into fixed points characterized by functions $\tilde{Z}^*(\phi)$ and $\tilde{U}^*(\phi)$, which depend on the parameters u and K (Fig. 2). While $\tilde{U}^*(\phi)$ slightly deviates from the cosine form of the initial potential $\tilde{U}_{k_{\rm in}}(\phi) = -(u/k_{\rm in}^2)\cos(\beta\phi)$, we see that $\tilde{Z}^*(\phi)$ acquires a strong dependence on ϕ. Z_k diverges as k^{-2} and the running Luttinger parameter $K_k \sim k^2$ vanishes for $k \to 0$.

Benchmarking: soliton and breather masses.—The smallest excitation gap M of the quantum sine-Gordon model corresponds to the inverse correlation length of the two-dimensional classical model [Eq. (3)],

$$M^2 = \lim_{k\to 0}\frac{U_k''(0)}{Z_k(0)} = \lim_{k\to 0} k^2\frac{\tilde{U}_k{}''(0)}{\tilde{Z}_k(0)}. \quad (7)$$

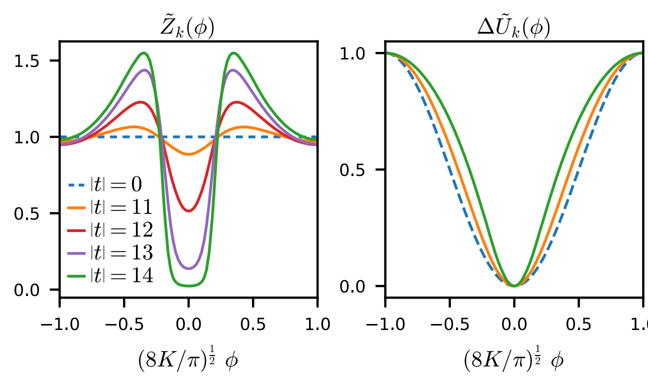

FIG. 2. $\tilde{Z}_k(\phi)$ and $\tilde{U}_k(\phi)$ for various values of $t = \ln(k/k_{\rm in})$. $\Delta\tilde{U}_k(\phi)$ is given by $\tilde{U}_k(\phi) - \tilde{U}_k(0)$ normalized so that $\Delta\tilde{U}_k(\pm\sqrt{\pi/8K}) = 1$. In Figs. 2 and 3, $\Lambda = 1$ and $u/\Lambda^2 = 10^{-3}$.

Since $\tilde{U}_k''(0)$ converges to a finite value (this property is preserved even if one retains only the first harmonics of $\tilde{U}_k(\phi)$) $\tilde{Z}_k(0)$ must vanish as k^2 for M to take a nonzero value in the massive phase. $\tilde{Z}_k(\phi)$ being a normalized function, $\langle \tilde{Z}_k(\phi)\rangle_\phi = 1$, this is possible only if $\tilde{Z}_k(\phi)$ strongly varies with ϕ. Thus only a functional approach where the coefficient of $(\boldsymbol{\nabla}\phi)^2$ in the effective action is a function $Z_k(\phi)$, and not a mere ϕ-independent number, can predict the mass of the lowest excitation. Numerically we observe a rapid convergence of $k^2\tilde{U}_k''(0)/\tilde{Z}_k(0)$ when $k \to 0$ in agreement with a previous study [27].

Only excitations that are in the same topological sector as the ground state, namely $Q = 0$, contribute to the mass M [1]. The lowest excitation in this sector is a soliton-antisoliton pair with mass

$$2M_{\rm sol} = b\Lambda \frac{4\Gamma(\frac{K}{2-2K})}{\sqrt{\pi}\Gamma(\frac{1}{2-2K})}\left[\frac{\Gamma(1-K)}{\Gamma(K)}\frac{\pi u}{2(b\Lambda)^2}\right]^{1/(2-2K)}, \qquad (8)$$

when $1/2 \le K \le 1$ [$M_{\rm sol}$ is the mass of a single (anti) soliton] and a breather with mass

$$M_1 = 2M_{\rm sol}\sin\left(\frac{\pi}{2}\frac{K}{1-K}\right), \qquad (9)$$

when $0 \le K < 1/2$ [36]. Here b is a scale parameter that depends on the precise implementation of the UV cutoff Λ in Eq. (1). Figure 3 shows the value of M obtained from FRG (we refer to the Supplemental Material [30] for a discussion of the implementation of the UV cutoff Λ and the determination of the scale factor b). For $0 \le K \le 0.4$ our result for the breather mass $M \equiv M_1$ deviates from the exact value by at most 2%. The agreement becomes nearly perfect for $K \ll 0.4$, which is due to the fact that the initial value $\Gamma^{(2)}_{k_{\rm in}}(\mathbf{q}, \phi = 0) = \mathbf{q}^2 + 8\pi K u$ gives the exact breather

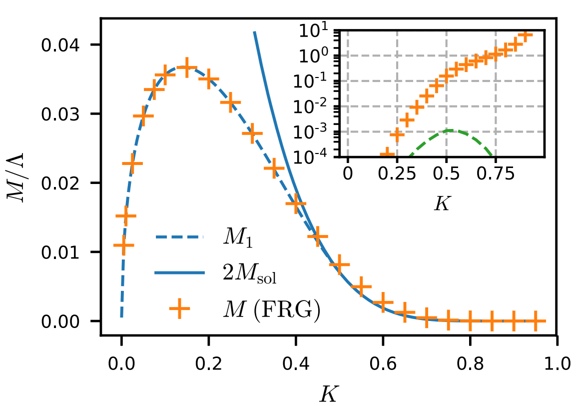

FIG. 3. Mass M of the lowest excitation as obtained from the FRG approach. The solid and dashed lines show the exact values of $2M_{\rm sol}$ and M_1 (the latter being defined only for $K \le 1/2$) [Eqs. (8) and (9)]. The inset shows the relative (crosses) and absolute (dashed line) errors of the FRG result.

mass $M_{1,\rm cl} = \sqrt{8\pi K u}$ in the semiclassical limit $K \to 0$ [30]. For $0.4 \le K \le 1$ the agreement between M and the exact value $2M_{\rm sol}$ is not as good and varies from ~2% for $K \simeq 0.4$ to more than 100% for K near 1. Note however that M goes to zero when $K \to 1$ and the absolute error remains below 10^{-3} for all values of K (see the inset in Fig. 3). In the immediate vicinity of $K = 1$, the behavior of the mass M differs from $2M_{\rm sol}$ [Eq. (8)] and one recovers the standard BKT scaling characterized by an essential singularity of the correlation length [8].

The Lukyanov-Zamolodchikov conjecture.—To obtain the expectation value of the exponential fields, we consider the partition function [Eq. (3)] in the presence of an external source term $\int d^2r(h^* e^{i\sqrt{8\pi}\alpha\varphi} + \text{c.c.})$ so that $\langle e^{i\sqrt{8\pi}\alpha\varphi(\mathbf{r})}\rangle$ can be obtained from $\ln \mathcal{Z}_k[J, h^*, h]$ by functional differentiation with respect to $h^*(\mathbf{r})$. To the second order of the derivative expansion, the effective action now reads

$$\Gamma_k[\phi, h^*, h] = \int d^2r\left\{\frac{1}{2}Z_k(\phi, h^*, h)(\boldsymbol{\nabla}\phi)^2 + U_k(\phi, h^*, h)\right\} \qquad (10)$$

and

$$\langle e^{i\sqrt{8\pi}\alpha\varphi(\mathbf{r})}\rangle = -\frac{\partial U_k(\phi = 0, h^*, h)}{\partial h^*}\bigg|_{h^*=h=0}. \qquad (11)$$

From the flow equation of $\Gamma_k[\phi, h^*, h]$ we obtain two coupled equations for $U_k^{(10)}(\phi) \equiv \partial_{h^*}U_k(\phi, h^*, h)|_{h^*=h=0}$ and $Z_k^{(10)}(\phi) \equiv \partial_{h^*}Z_k(\phi, h^*, h)|_{h^*=h=0}$ with initial conditions $U_{k_{\rm in}}^{(10)}(\phi) = -e^{i\sqrt{8\pi}\alpha\phi}$ and $Z_{k_{\rm in}}^{(10)}(\phi) = 0$ [30].

We have computed the expectation value of the exponential fields $e^{in\sqrt{2\pi K}\varphi}$ (n integer). These are the natural fields to consider in the sine-Gordon model. For instance, in one-dimensional quantum fluids, they arise from products of single-particle fields. The FRG results for $1 \le n \le 5$ are shown in Fig. 4.

For $n = 1$ we find an excellent agreement between the FRG results and the conjecture, with a difference $\epsilon_{\rm abs}$ well below 0.01 for all values of K. The relative disagreement $\epsilon_{\rm rel}$ is small for $K < 0.5$ but increases for larger values of K and becomes of order of 100% for K near 1. For these values of K, however, the expectation value $\langle e^{i\sqrt{2\pi K}\varphi}\rangle$ is very small and what matters is $\epsilon_{\rm abs}$.

Note that the Lukyanov-Zamolodchikov conjecture breaks down in the vicinity of $K = 1/n$ since the expectation value $\langle e^{in\sqrt{2\pi K}\varphi}\rangle$ given by Eq. (2) diverges when $K \to 1/n$ [37]. This explains the steep upturn near $K = 1/n$ of the lines showing the conjecture in Fig. 4. Decreasing the value of u/Λ^2 confines the upturn more and more to the vicinity of $1/n$.

For $n \ge 2$, $\epsilon_{\rm rel}$ behaves similarly to the case $n = 1$ but $\epsilon_{\rm abs}$ is also a monotonically increasing function of K (see

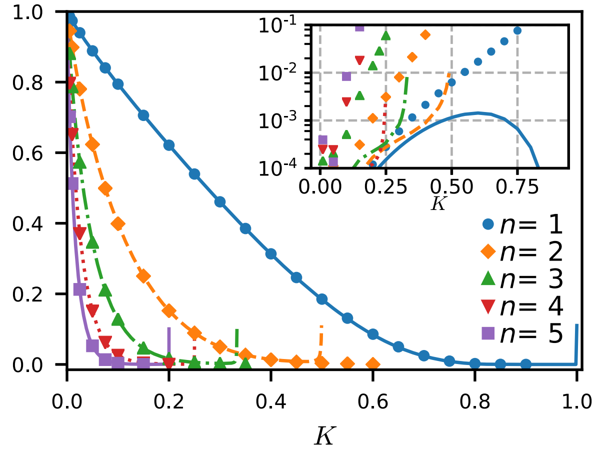

FIG. 4. Expectation value $\langle e^{in\sqrt{2\pi K}\varphi}\rangle$ as obtained from FRG (symbols) vs the Lukyanov-Zamolodchikov conjecture [Eq. (2)] valid for $K < 1/n$ (lines). The inset shows the relative (symbols) and absolute (lines) disagreements between the FRG results and the conjecture, respectively, $\epsilon_{\rm rel}$ and $\epsilon_{\rm abs}$. ($\Lambda = 1$ and $u/\Lambda^2 = 10^{-4}$).

the inset in Fig. 4). $\epsilon_{\rm abs}$ remains nevertheless below 0.01 up to values of K very close to $1/n$; for $u/\Lambda^2 = 10^{-4}$ this is the case for $K = 0.49$ (and $n = 2$), $K = 0.33$ ($n = 3$), and $K = 0.248$ ($n = 4$). Moreover $\epsilon_{\rm rel}$ decreases when u/Λ^2 is reduced (which extends the domain of validity of the conjecture to higher values of K, i.e., to values of K closer to $1/n$). For instance, for $n = 2$ and $K = 0.49$, we find $\epsilon_{\rm rel} = 78/72/66\%$ (while $\epsilon_{\rm abs} = 0.097/0.0097/0.00098$) for $u/\Lambda^2 = 10^{-3}/10^{-4}/10^{-5}$. We therefore ascribe the apparent disagreement between the FRG results and the conjecture near $K = 1/n$ to the breakdown of the latter when $K \to 1/n$. In fact the change of concavity in the curves showing $\epsilon_{\rm abs}$ in the inset of Fig. 4 suggests that the conjecture might deviate from the correct result well before $K = 1/n$ (e.g., $K \sim 0.4$ for $n = 2$ and $u/\Lambda^2 = 10^{-4}$). A conservative estimate is that the FRG reproduces Eq. (2), in the domain of validity of the conjecture, to an accuracy (defined as the minimum of $\epsilon_{\rm abs}$ and $\epsilon_{\rm rel}$) better than 0.01.

Conclusion.—Contrary to the perturbative RG [23,38,39], which correctly predicts the phase diagram of the quantum sine-Gordon model but fails to describe the massive phase, the nonperturbative FRG allows us to continue the flow into the strong-coupling regime and thus compute the low-energy properties of the massive phase. The fact that FRG captures genuinely nonperturbative topological excitations, namely (anti)solitons and breathers, proves its efficiency and is reminiscent of its ability to describe most universal properties of the BKT transition in the linear O(2) model [40–42] for which it is widely admitted that topological defects (vortices) play a crucial role.

The FRG result for the expectation value $\langle e^{in\sqrt{2\pi K}\varphi}\rangle$ of the exponential fields is in very good agreement with the conjecture proposed by Lukyanov and Zamolodchikov [15]. The minimum of the relative and absolute disagreements is smaller than 0.01 for all values of n except in the immediate vicinity of $K = 1/n$ where the conjecture breaks down. This undoubtedly provides us with a very strong support of the Lukyanov-Zamolodchikov conjecture. We also stress that FRG allows one to obtain $\langle e^{in\sqrt{2\pi K}\varphi}\rangle$ for all values of K whereas the conjecture is limited to the range $K < 1/n$.

Finally we would like to point out that the nonperturbative FRG approach presented in this Letter opens up the possibility to study various nonintegrable extensions of the quantum sine-Gordon model where both perturbative RG and exacts methods are inoperative in the strong-coupling phase.

We thank P. Azaria for enlightening discussions and a critical reading of the manuscript.

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